

Group Representations and High-Resolution Central Limit Theorems for Subordinated Spherical Random Fields

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Abstract

We study the weak convergence (in the high-frequency limit) of the frequency components associated with Gaussian-subordinated, spherical and isotropic random fields. In particular, we provide conditions for asymptotic Gaussianity and we establish a new connection with random walks on the hypergroup $\widehat{SO}(3)$ (the dual of $SO(3)$), which mirrors analogous results previously established for fields defined on Abelian groups (see Marinucci and Peccati (2007)). Our work is motivated by applications to cosmological data analysis, and specifically by the probabilistic modelling and the statistical analysis of the Cosmic Microwave Background radiation, which is currently at the frontier of physical research. To obtain our main results, we prove several fine estimates involving convolutions of the so-called *Clebsch-Gordan coefficients* (which are elements of unitary matrices connecting reducible representations of $SO(3)$); this allows to interpret most of our asymptotic conditions in terms of coupling of angular momenta in a quantum mechanical system. Part of the proofs are based on recently established criteria for the weak convergence of multiple Wiener-Itô integrals.

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1 Introduction

This paper deals with weak limit theorems involving the high-frequency components (in the sense of the spherical harmonics decomposition) of random fields defined on the unit sphere \mathbb{S}^2 . Our results are motivated by a number of mathematical issues arising in connection with the probabilistic and statistical analysis of the Cosmic Microwave Background radiation (see e.g. [6]). We start by giving a description of our abstract mathematical framework, along with a sketch of the main results of the paper; the subsequent Section 1.2 focuses on the physical motivations and applications of our research. Here, and for the rest of the paper, all random elements are defined on a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

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1.1 General framework and outline of the main results

We shall consider real-valued random fields $\{\tilde{T}(x) : x \in \mathbb{S}^2\}$ enjoying the following properties:

$$\mathbb{E}\tilde{T}(x) = 0, \quad \mathbb{E}\tilde{T}^2(x) < +\infty \quad \text{and} \quad \tilde{T}(gx) \stackrel{\text{law}}{=} \tilde{T}(x), \quad (1)$$

for all $x \in \mathbb{S}^2$ and all $g \in SO(3)$, where $\stackrel{\text{law}}{=}$ denotes equality in law (in the sense of stochastic processes). A field verifying the last relation in (1) is usually called *isotropic* or *rotationally-invariant* (in law). It is a standard result that the following spectral representation holds in the mean-square sense:

$$\tilde{T}(x) = \sum_{l=0}^{\infty} \tilde{T}_l(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm} Y_{lm}(x), \quad (2)$$

where $\{Y_{lm} : l \geq 0, m = -l, \dots, l\}$ is the collection of the spherical harmonics, and the $\{a_{lm}\}$ are the associated (harmonic) Fourier coefficients. For $l \geq 0$, we also write $C_l \triangleq \mathbb{E}|a_{lm}|^2$, and we call the sequence $\{C_l : l \geq 0\}$ the *angular power spectrum* of the random field \tilde{T} (note that C_l does not depend on m – see e.g. [2]). For every $l \geq 0$, the field \tilde{T}_l provides the projection of \tilde{T} on the subspace of $L^2(\mathbb{S}^2, dx)$ spanned by the class $\{Y_{lm} : m = -l, \dots, l\}$. The spherical harmonics form an orthonormal basis of $L^2(\mathbb{S}^2, dx)$ which can be derived from the restriction to the sphere of harmonic polynomials. In particular, in spherical coordinates $x = (\theta, \varphi)$ they can be written explicitly as: $Y_{00} \equiv 1/\sqrt{4\pi}$ and

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos \theta) e^{im\varphi}, \quad m \geq 0, \quad (3)$$

$$Y_{lm}(\theta, \varphi) = (-1)^m \overline{Y_{l,-m}}(\theta, \varphi), \quad m < 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi, \quad (4)$$

where, for $l \geq 1$ and $m = 0, 1, 2, \dots, l$, $P_{lm}(\cdot)$ denotes the Legendre polynomial of index l, m , i.e.,

$$P_{lm}(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \quad P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2-1)^l. \quad (5)$$

For a discussion of these and other properties of the spherical harmonics see e.g. [13, Chapter 9], or [28, Chapter 5]. For $l \geq 0$, the real-valued field \tilde{T}_l is called the *lth frequency component* of \tilde{T} . The expansion (2) can be achieved by many different routes, for instance by a Karhunen-Loève argument or by means of the stochastic Peter-Weyl theorem, see for instance [1], [3], [12] and [22]. The random harmonic coefficients $\{a_{lm}\}$ appearing in (2) form a triangular array of zero-mean random variables, which are complex-valued for $m \neq 0$ and such that $\mathbb{E}a_{lm} \overline{a_{l'm'}} = \delta_l^{l'} \delta_m^{m'} C_l$ (the bar denotes complex conjugation and δ is Kronecker's symbol; note also that $a_{lm} = (-1)^m \overline{a_{l,-m}}$). For a Gaussian random field \tilde{T} verifying (1), it is trivial that the set $\{a_{lm}\}$ is itself a complex-Gaussian array, with independent elements for $m \geq 0$. It is a simple but interesting fact that the converse also holds, i.e. that, under an isotropy assumption on \tilde{T} , the independence of the a_{lm} 's for $m \geq 0$ implies Gaussianity, see [2]. Apart from this result, the behaviour of the array $\{a_{lm}\}$ and of the projections $\{\tilde{T}_l\}$ for non-Gaussian isotropic fields is so far almost completely unexplored and open for research, although such objects are highly relevant for cosmological applications (see the next subsection). It should be stressed that the coefficients $\{a_{lm}\}$ depend on the choice of coordinates and are not intrinsic to the field, although their law is. In this sense, it is sometimes physically more sound to focus on the behaviour of the sequence of projections $\{\tilde{T}_l\}$, which are indeed invariant with respect to the choice of coordinates.

In what follows, we focus on non-Gaussian fields \tilde{T} that are *Gaussian-subordinated*, and we address the previous topic by studying the asymptotic behaviour of $\{a_{lm}\}$ and $\{\tilde{T}_l\}$, as $l \rightarrow +\infty$. Recall that \tilde{T} is called *Gaussian-subordinated* whenever $\tilde{T}(x) = F(T(x))$, where F is a suitable real-valued function, and T is an isotropic spherical (real) Gaussian field. In particular, our purpose is to establish sufficient (and sometimes, also necessary) conditions on F and on the law of T to have that the following two phenomena take place: **(I)** as $l \rightarrow +\infty$, for a fixed m and for an appropriate sequence $\tau_1(l)$ ($l \geq |m|$), the sequence

$$\tau_1(l) \times a_{lm} = \tau_1(l) \int_{\mathbb{S}^2} F(T(z)) \overline{Y_{lm}(z)} dz, \quad l \geq |m|$$

converges in law to a Gaussian random variable (real-valued for $m = 0$, and complex-valued for $m \neq 0$); **(II)** for a suitable real-valued sequence $\tau_2(l)$ ($l \geq 0$) and for l sufficiently large, the finite-dimensional distributions of the field

$$\tau_2(l) \times \tilde{T}_l(\cdot) = \tau_2(l) \sum_{m=-l, \dots, l} a_{lm} Y_{lm}(\cdot),$$

are close (for instance, in the sense of the Prokhorov distance – see [21]) to those of a real spherical Gaussian field. Note that both results **(I)** and **(II)** can be interpreted as CLTs *in the high-frequency* (or *high-resolution*) *sense*, since they involve Gaussian approximations and are established by letting the frequency index l diverge to infinity.

Our findings generalize previous results, obtained in [17], for fields defined on Abelian compact groups. One of our main tools is a result concerning the Gaussian approximation of multiple Wiener-Itô integrals established in [21] (see also [19], [23] and [24]). These CLTs can be seen as a simplification of the combinatorial *method of diagrams and cumulants* (see e.g. [26]). These techniques, combined with the use of group representation theory, lead to one of the main contributions of this paper: the derivation of sufficient (or necessary and sufficient) conditions for **(I)** and **(II)**, expressed in terms of convolutions of *Clebsch-Gordan coefficients* (see e.g. [28, Ch. 4]), which are the elements of unitary matrices connecting specific reducible representations of $SO(3)$. Clebsch-Gordan coefficients are widely used in quantum mechanics, and admit a well-known interpretation in terms of probability amplitudes related to the coupling of angular momenta in a quantum mechanical system (see [13], [28] or Sections 3 and 6 below). We will also show that many of our conditions can be alternatively restated in terms of ‘bridges’ of random walks on $\widehat{SO(3)}$ (the dual of $SO(3)$). The definition of such random walks differs from the classic one given in [8], although the two approaches can be related through the notion of *mixed quantum state* (see Section 6). Note that an analogous connection with random walks on \mathbb{Z}^d was pointed out in [17].

1.2 Cosmological motivations

The Cosmic Microwave Background radiation (hereafter CMB) can be viewed as a relic radiation of the Big Bang, providing maps of the primordial Universe before the formation of any of the current structures (approximately, 3×10^5 years after the Big Bang); as such, it is acknowledged as a goldmine of information for fundamental physics. Many satellite experiments involving hundred of physicists throughout the world are devoted to the construction of spherical maps of the CMB radiation, and for pioneering work in this area G. Smoot and J. Mather were awarded

the Nobel Prize for Physics in 2006 – see for instance <http://map.gsfc.nasa.gov/> for more details.

The crucial point is that most cosmological models imply that the CMB radiation is the realization of a random field $\{\tilde{T}(x) : x \in \mathbb{S}^2\}$, verifying the three conditions in (1); each $x \in \mathbb{S}^2$ corresponds to a direction in which the CMB radiation is measured. The isotropic property can be seen as a consequence of Einstein’s *cosmological principle*, roughly stating that, on sufficiently large distance scales, the Universe looks identical everywhere in space (homogeneity) and appears the same in every direction (isotropy). A central issue in modern cosmology relates therefore to the distribution of the CMB random field \tilde{T} , which is predicted to be (close to) Gaussian by some models for the dynamics at primordial epochs (for instance, by the so-called *inflationary scenario*), and non-Gaussian by other models, where fluctuations are generated by topological defects arising in phase transitions of a thermodynamical nature – see for instance [6]. Many testing procedures have been proposed to tackle this issue; in some form, they all rely asymptotically on the behaviour of the field at the highest frequencies (see for instance [4], [14] and the references therein). This is a sort of unescapable, foundational issue in Cosmology. By definition, the latter is a science based on a single realization, e.g. our Universe or the trace of its primordial structure in the form of the CMB radiation, which is observed at higher and higher resolutions. As such, an asymptotic theory for statistical tests is possible only in the sense of observations at higher and higher frequencies (smaller and smaller scales) becoming available as the experiments become more sophisticated. In particular, any satellite experiment measuring the CMB radiation can reconstruct the spherical harmonic development appearing in (2) only up to a finite frequency l_{\max} , the quantity π/l_{\max} representing approximately the *angular resolution* of the experiment (the pioneering satellite COBE (1993) could reach a frequency $l_{\max} \simeq 20$, WMAP (2003, 2006) improved this limit to $l_{\max} \simeq 600/800$, and Planck (to be launched in 2008) is expected to reach $l_{\max} \simeq 2500/3000$). In order for such procedures to yield consistent outcomes, one should therefore figure out what is the limiting behaviour of $\{\tilde{T}_l\}$, for $l \gg 0$, under different distributional assumptions on \tilde{T} . Some Monte Carlo evidence (see for instance [16] and the references therein) has suggested that this behaviour may be close to Gaussian even in circumstances where the underlying field \tilde{T} clearly is not. The investigation of this issue is necessary for rigorous inference on CMB data, and in particular for non-Gaussianity tests. The relevance of the asymptotic behaviour of the $\{\tilde{T}_l\}$, however, goes much beyond the issue of such tests, and relates indeed to the whole statistical analysis of CMB – which is largely dominated by likelihood approaches (see [7]).

We stress by now that the results we provide cover models that are quite relevant for cosmological applications, for instance the so called *Sachs-Wolfe model*, which represents the standard starting model for the inflationary scenario (see for instance [4], [6]). In its simplest version, this model implies that the CMB is a straightforward quadratic transformation of an underlying Gaussian field, i.e.

$$\tilde{T}(x) = T(x) + f_{NL} \{T(x)^2 - \mathbb{E}T(x)^2\} \quad , \quad x \in \mathbb{S}^2, \quad (6)$$

where f_{NL} is a nonlinearity parameter depending on constants from particle physics and T is Gaussian and isotropic. As a special case, our results do allow for a complete characterization of the high-frequency behaviour of models such as (6), and in this sense they are immediately applicable in the cosmological literature.

1.3 Plan

In Section 2 we provide some background material on isotropic random fields on the sphere. Section 3 is devoted to a discussion on representation theory for the group of rotations $SO(3)$ and the so-called Clebsch-Gordan coefficients, which will play a crucial role in the analysis to follow. In Section 4 we state and prove a general CLT result for the spherical harmonics coefficients and the high-frequency components of a field arising from polynomial transformations of arbitrary order of a subordinating Gaussian process. In Section 5 we provide a more detailed analysis of necessary and sufficient condition for the CLT to hold in the case of quadratic and cubic transformations; we also highlight the connections between our conditions and the theory of random walks on hypergroups. The interplay with random walks on hypergroups is further explored in Section 6, where some comparisons with the existing literature are provided, and some physical interpretations of our conditions in terms of randomly interacting quantum particles are given. In Section 7, we turn our attention to more explicit conditions on the angular power spectrum, and we discuss an exponential/algebraic duality which parallels to some extent some earlier findings in the Abelian case.

2 Preliminaries on Gaussian and Gaussian-subordinated isotropic fields

As in the Introduction, we denote by \mathbb{S}^2 the unit sphere $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$. For every rotation $g \in SO(3)$ and every $x \in \mathbb{S}^2$, the symbol gx indicates the canonical action of g on x (see [28, Ch. 1], as well as Section 3 below, for further details). We will systematically write dx for the Lebesgue measure on \mathbb{S}^2 , and we denote by $L^2(\mathbb{S}^2, dx)$ the class of complex-valued functions on \mathbb{S}^2 which are square-integrable with respect to dx . We denote by $\{Y_{lm} : l \geq 0, m = -l, \dots, l\}$ the basis of $L^2(\mathbb{S}^2, dx)$ given by spherical harmonics, as defined via (3) and (4). From now on, we shall denote by $T = \{T(x) : x \in \mathbb{S}^2\}$ a centered, real-valued and *Gaussian* random field parametrized by \mathbb{S}^2 . We also suppose that T is *isotropic*, that is, for every $g \in SO(3)$ one has that $T(x) \stackrel{\text{law}}{=} T(gx)$, where the equality holds in the sense of finite dimensional distributions. To simplify the notation, we also assume that $\mathbb{E}T(x)^2 = 1$. Following e.g. [2] (but see also [3], [22] and [25]), one deduces from isotropy that T admits the spectral decomposition

$$T(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm;1} Y_{lm}(x) = \sum_{l=0}^{\infty} T_l(x), \quad x \in \mathbb{S}^2, \quad (7)$$

where $a_{lm;1} \triangleq \int_{\mathbb{S}^2} T(x) \overline{Y_{lm}(x)} dx$ (the role of the subscript “ $lm;1$ ” will be clarified in the following discussion), $T_l(x) \triangleq \sum_{m=-l}^l a_{lm;1} Y_{lm}(x)$, and the convergence takes place in $L^2(\mathbb{P})$ for every fixed x , as well as in $L^2(\mathbb{P} \otimes dx)$. The next result gives a simple and very useful characterization of the joint law of the complex-valued array $\{a_{lm;1} : l \geq 0, m = -l, \dots, l\}$. For every $z \in \mathbb{C}$, the symbols $\Re(z)$ and $\Im(z)$ indicate, respectively, the real and the imaginary part of z .

Proposition 1 *Let T be the centered, isotropic and Gaussian random field appearing in (7). Then: (i) for every $l \geq 0$ the random variable $a_{l0;1}$ is real-valued, centered and Gaussian; (ii) for every $l \geq 1$, and every $m = 1, \dots, l$, the random variable $a_{lm;1}$ is complex-valued and such that $a_{lm;1} = (-1)^m \overline{a_{l-m;1}}$, and moreover $\mathbb{E}(\Re(a_{lm;1})^2) = \mathbb{E}(\Im(a_{lm;1})^2) = \mathbb{E}(a_{l0;1}^2)/2 = C_l/2$, for*

some constant $C_l \in [0, +\infty)$ not depending on m , and

$$\mathbb{E}(\Re(a_{lm;1}) \Im(a_{lm;1})) = 0; \quad (8)$$

(iii) for every $l \geq 1$ and every $m = -l, \dots, l$, the random coefficient $a_{lm;1}$ is independent of $a_{l'm';1}$ for every $l' \geq 0$ such that $l' \neq l$ and every $m' = -l', \dots, l'$. By noting $C_0 \triangleq \mathbb{E}(a_{00;1}^2)$, one also has the relation

$$1 = \mathbb{E}[T(x)^2] = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} C_l. \quad (9)$$

The reader is referred to [2] for a proof of Proposition 1, as well as for several converse statements. Here, we shall only stress that formula (9) is a consequence of the well-known relation (see e.g. [28])

$$\sum_{m=-l}^l Y_{lm}(x) \overline{Y_{lm}(y)} = \frac{2l+1}{4\pi} P_l(\cos \langle x, y \rangle), \quad x, y \in \mathbb{S}^2, \quad (10)$$

where $\langle x, y \rangle$ is the angle between x and y . Observe that property (8) implies that $\Re(a_{lm;1})$ and $\Im(a_{lm;1})$ are independent centered Gaussian random variables. Moreover, the combination of (8) and point (iii) in the statement of Proposition 1 yields that $\mathbb{E}(a_{lm;1} \overline{a_{l'm';1}}) = 0$, $\forall (l, m) \neq (l', m')$. Finally, it is also evident that points (i)-(iii) in the previous statement imply that the law of an isotropic Gaussian field such as T is completely characterized by its angular power spectrum $\{C_l : l \geq 0\}$. To avoid trivialities, we will always work under the following assumption:

Assumption. The angular power spectrum $\{C_l : l \geq 0\}$ is such that $C_l > 0$ for every l .

Note that the results of this paper could be extended without difficulties (but at the cost of an heavier notation) to the case of a power spectrum such that $C_l \neq 0$ for infinitely many l 's. In the subsequent sections, we shall obtain high-frequency CLTs for centered isotropic spherical fields that are *subordinated* to the Gaussian field T defined above.

Definition A (*Subordinated fields*). Let $L_0^2(\mathbb{R}, e^{-z^2/2} dz)$ indicate the class of real-valued functions $F(z)$ on \mathbb{R} , which are square-integrable with respect to the measure $e^{-z^2/2} dz$ and such that $\int F(z) e^{-z^2/2} dz = 0$. A (centered) random field $\tilde{T} = \{\tilde{T}(x) : x \in \mathbb{S}^2\}$ is said to be *subordinated* to the Gaussian field T appearing in (2) if there exists $F \in L_0^2(\mathbb{R}, e^{-z^2/2} dz)$ such that $\tilde{T}(x) = F[T](x)$, $\forall x \in \mathbb{S}^2$, where the symbol $F[T](x)$ stands for $F(T(x))$. Whenever \tilde{T} is subordinated, we will rather use the notation $F[T](x)$ instead of $\tilde{T}(x)$, in order to emphasize the role of the function F . Of course, if $F(z) = z$, then $F[T](x) = \tilde{T}(x) = T(x)$.

It is immediate to check that, since T is isotropic, a subordinated field $F[T](\cdot)$ as in Definition A is necessarily isotropic. As a consequence, following again [2] or [22], one deduces that $F[T]$ admits the spectral representation

$$F[T](x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l a_{lm}(F) Y_{lm}(x) = \sum_{l=0}^{\infty} F[T]_l(x), \quad x \in \mathbb{S}^2, \quad (11)$$

with convergence in $L^2(\mathbb{P})$ (for fixed x) and in $L^2(\Omega \times \mathbb{S}^2, \mathbb{P} \otimes dx)$. Here,

$$a_{lm}(F) \triangleq \int_{\mathbb{S}^2} F[T](y) \overline{Y_{lm}(y)} dy, \text{ and} \quad (12)$$

$$F[T]_l(x) \triangleq \sum_{m=-l}^l a_{lm}(F) Y_{lm}(x). \quad (13)$$

The complex-valued array $\{a_{lm}(F) : l \geq 0, m = -l, \dots, l\}$ always enjoys the following properties **(a)-(c)**: **(a)** for every $l \geq 0$, the random variable $a_{l0}(F)$ is real-valued, centered and Gaussian; **(b)** for every $l \geq 1$, and every $m = 1, \dots, l$, the random variable $a_{lm}(F)$ is complex-valued and such that

$$\begin{aligned} a_{lm}(F) &= (-1)^m \overline{a_{l-m}(F)} \quad ; \quad \mathbb{E}(\Re(a_{lm}(F)) \Im(a_{lm}(F))) = 0 \\ \mathbb{E}(\Re(a_{lm}(F))^2) &= \mathbb{E}(\Im(a_{lm}(F))^2) = \mathbb{E}(a_{l0}(F)^2)/2 = C_l(F)/2, \end{aligned}$$

where the finite constant $C_l(F) \geq 0$ depends solely on F and l ; **(c)** $\mathbb{E}(a_{lm}(F) \times \overline{a_{l'm'}(F)}) = 0$, $\forall (l, m) \neq (l', m')$. Note that, in general, it is no longer true that $\Re(a_{lm}(F))$ and $\Im(a_{lm}(F))$ are independent random variables. Moreover, we state the following consequence of [2, Th. 7]: *for every $l \geq 1$, the coefficients $(a_{l0}(F), \dots, a_{ll}(F))$ are stochastically independent if, and only if, they are Gaussian.* Also, $\mathbb{E}(F[T](x)^2) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} C_l(F)$.

In the subsequent sections, a crucial role will be played by the class of *Hermite polynomials*. Recall (see e.g. [11, p. 20]) that the sequence $\{H_q : q \geq 0\}$ of Hermite polynomials is defined by the differential relation

$$H_q(z) = (-1)^q e^{\frac{z^2}{2}} \frac{d^q}{dz^q} e^{-\frac{z^2}{2}}, \quad z \in \mathbb{R}, \quad q \geq 0; \quad (14)$$

it is well-known that the sequence $\{(q!)^{-1/2} H_q : q \geq 0\}$ defines an orthonormal basis of the space $L^2(\mathbb{R}, (2\pi)^{-1/2} e^{-z^2/2} dz)$. When a subordinated field has the form (for $q \geq 2$) $H_q[T](x)$, $x \in \mathbb{S}^2$ (that is, when $F = H_q$ in Definition A), we will use the shorthand notation:

$$T^{(q)}(x) \triangleq H_q[T](x), \quad x \in \mathbb{S}^2, \quad (15)$$

$$a_{lm;q} \triangleq a_{lm}(H_q), \quad (16)$$

$$T_l^{(q)}(x) \triangleq H_q[T]_l(x), \quad l \geq 1, x \in \mathbb{S}^2, \quad (17)$$

$$\overline{T}_l^{(q)}(x) \triangleq \text{Var}\left(T_l^{(q)}(x)\right)^{-1/2} T_l^{(q)}(x), \quad l \geq 1, x \in \mathbb{S}^2, \quad (18)$$

$$\tilde{C}_l^{(q)} \triangleq C_l(H_q) = \mathbb{E}|a_{lm;q}|^2, \quad l \geq 1, m = -l, \dots, l. \quad (19)$$

To justify our notation (15)–(19), we recall that for every fixed x the random variable $H_q[T](x) = H_q(T(x))$ is just the q th Wick power of $T(x)$ (see for instance [11]). We conclude the section with an easy Lemma, that will be used in Section 4.

Lemma 2 *Let $F[T](x)$, $x \in \mathbb{S}^2$, be an (isotropic) subordinated field as in Definition A. Then, for every $l \geq 1$ one has the following:*

1. *The random field $x \mapsto F[T]_l(x)$ defined in (13) is real-valued and isotropic;*
2. *For every fixed $x \in \mathbb{S}^2$, $F[T]_l(x) \stackrel{\text{law}}{=} \sqrt{\frac{2l+1}{4\pi}} a_{l0}(F)$, where the coefficient $a_{l0}(F)$ is defined according to (12), and consequently $\mathbb{E}(F[T]_l(x)^2) = \frac{2l+1}{4\pi} C_l(F)$;*
3. *The normalized random field*

$$\overline{F[T]_l}(x) = \left[\frac{(2l+1) C_l(F)}{4\pi} \right]^{-1/2} F[T]_l(x) \quad (20)$$

has a covariance structure given by: for every $x, y \in \mathbb{S}^2$,

$$\mathbb{E} \left(\overline{F[T]_l}(x) \times \overline{F[T]_l}(y) \right) = P_l(\cos \langle x, y \rangle), \quad (21)$$

where $P_l(\cdot)$ is the l th Legendre polynomial defined in (5) and, as before, $\langle x, y \rangle$ is the angle between x and y .

Proof. Point 1. is straightforward. To prove point 2. define (in polar coordinates) $x_0 = (0, 0)$ and use the isotropy property stated at point 1. to write

$$F[T]_l(x) \stackrel{\text{law}}{=} F[T]_l(x_0) = \sum_{m=-l}^l a_{lm}(F) Y_{lm}(x_0) = \sqrt{\frac{2l+1}{4\pi}} a_{l0}(F),$$

since (3) implies that $Y_{lm}(x_0) = \sqrt{(2l+1)/4\pi} \delta_m^0$. Finally, to prove relation (21) we use (10) to deduce that, for every $x, y \in \mathbb{S}^2$,

$$\mathbb{E}(F[T]_l(x) F[T]_l(y)) = C_l(F) \frac{2l+1}{4\pi} P_l(\cos \langle x, y \rangle),$$

thus giving the desired conclusion (recall that $P_l(1) = 1$). ■

For instance, a first consequence of Lemma 2 is that, for every $q \geq 2$,

$$\mathbb{E}(T_l^{(q)}(x)^2) = (2l+1) \tilde{C}_l^{(q)}/4\pi \quad (22)$$

where we used the notation introduced at (15)-(19), so that $\overline{T}_l^{(q)}(x) = [(2l+1) \tilde{C}_l^{(q)}/4\pi]^{-1/2} T_l^{(q)}(x)$.

The main aim of the subsequent sections is to provide an accurate solution to the following problems **(P-I)**–**(P-III)**.

(P-I) For a fixed $q \geq 2$, find conditions on the power spectrum $\{C_l : l \geq 0\}$ of T , to have that the subordinated process $T^{(q)} = \{T^{(q)}(x) : x \in \mathbb{S}^2\}$ defined in (15) is such that, for every $x \in \mathbb{S}^2$,

$$\sqrt{(2l+1) \tilde{C}_l^{(q)}/4\pi} \times T_l^{(q)}(x) \xrightarrow[l \rightarrow +\infty]{\text{law}} N, \quad (23)$$

where N is a centered standard Gaussian random variable.

(P-II) Under the conditions found at **(P-I)**, study the asymptotic behaviour, as $l \rightarrow +\infty$, of the vector

$$\sqrt{(2l+1) \tilde{C}_l^{(q)}/4\pi} \times \left(T_l^{(q)}(x_1), \dots, T_l^{(q)}(x_k) \right), \quad (24)$$

for every $x_1, \dots, x_k \in \mathbb{S}^2$.

(P-III) Combine **(P-I)** and **(P-II)** to study the asymptotic behaviour (in particular, the asymptotic Gaussianity), as $l \rightarrow +\infty$, of vectors of the type

$$\sqrt{(2l+1) C_l(F)/4\pi} \times (F[T]_l(x_1), \dots, F[T]_l(x_k)), \quad (25)$$

for every $x_1, \dots, x_k \in \mathbb{S}^2$ and every $F \in L_0^2(\mathbb{R}, e^{-z^2/2} dz)$.

Note that Problems **(P-I)**–**(P-III)** are stated in increasing order of generality. We observe also the following fact: since (21) holds, and since the limit of $P_l(\langle x, y \rangle)$ ($l \rightarrow +\infty$) does not exist

in general, it will not be possible to prove that the vectors in (24) and (25) converge in law to some Gaussian limit. However, by using the results developed in [21], we will be able to establish conditions under which the laws of such vectors are “asymptotically close” to a sequence of k -dimensional Gaussian distributions. As already mentioned, to study **(P-I)–(P-III)** we shall use estimates involving the so-called *Clebsch-Gordan* coefficients, that are elements of unitary matrices connecting some reducible representations of $SO(3)$. The definition and the analysis of some crucial properties of Clebsch-Gordan coefficients are the object of the next section.

3 A primer on Clebsch-Gordan coefficients

In this subsection, we need to review some basic representation theory results for $SO(3)$, the group of rotations in \mathbb{R}^3 . We refer the reader to standard textbooks (for instance, [28] and [29]) for further details, as well as for any unexplained notion or definition. It should be stressed that most of our arguments below could be extended to general compact groups with known representations; however, throughout the following we shall stick to the group of rotations $SO(3)$, mainly for the sake of notational simplicity.

We recall first that each element $g \in SO(3)$ can be parametrized by the set (α, β, γ) of so-called *Euler angles*, where $0 \leq \alpha < 2\pi$, $0 \leq \beta \leq \pi$ and $0 \leq \gamma < 2\pi$. In these coordinates, a complete set of irreducible matrix representations for $SO(3)$ is provided by the so-called *Wigner's D matrices* $D^l(\alpha, \beta, \gamma)$, of dimensions $(2l+1) \times (2l+1)$ for $l = 0, 1, 2, \dots$ – see [28, Ch. 4] for an analytic expression. Here, we simply point out that the elements of $D^l(\alpha, \beta, \gamma)$ are related to the spherical harmonics by the relationship

$$D_{m0}^l(\alpha, \beta, \gamma) = (-1)^m \sqrt{\frac{4\pi}{2l+1}} Y_{l-m}(\beta, \alpha) = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}^*(\beta, \alpha), \quad (26)$$

from which it is not difficult to show how the usual spectral representation for random fields on the spheres (for instance (2) and (7)) is really just the stochastic Peter-Weyl Theorem on $\mathbb{S}^2 = SO(3)/SO(2)$. The reader is referred e.g. to [29] and [27] for further discussions on the Peter-Weyl Theorem, and to [2], [3] and [22] for several related probabilistic results.

It follows from standard representation theory that we can exploit the family $\{D^l\}_{l=0,1,2,\dots}$ to build alternative (reducible) representations, either by taking the tensor product family $\{D^{l_1} \otimes D^{l_2}\}_{l_1, l_2}$, or by considering direct sums $\{\oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l\}_{l_1, l_2}$; these representations have dimensions $(2l_1+1)(2l_2+1) \times (2l_1+1)(2l_2+1)$ and are unitarily equivalent, whence there exists a unitary matrix $C_{l_1 l_2}$ such that

$$\{D^{l_1} \otimes D^{l_2}\} = C_{l_1 l_2} \left\{ \oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l \right\} C_{l_1 l_2}^*. \quad (27)$$

Here, $C_{l_1 l_2}$ is a $\{(2l_1+1)(2l_2+1) \times (2l_1+1)(2l_2+1)\}$ block matrix with blocks $C_{l_1(m_1)l_2}^l$ of dimensions $(2l_2+1) \times (2l_1+1)$, $m_1 = -l_1, \dots, l_1$. The elements of such a block are indexed by m_2 (over rows) and m (over columns). More precisely,

$$\begin{aligned} C_{l_1 l_2} &= \left[C_{l_1(m_1)l_2}^l \right]_{m_1=-l_1, \dots, l_1; l=|l_2-l_1|, \dots, l_2+l_1} \\ C_{l_1(m_1)l_2}^l &= \left\{ C_{l_1 m_1 l_2 m_2}^{lm} \right\}_{m_2=-l_2, \dots, l_2; m=-l, \dots, l}. \end{aligned}$$

The *Clebsch-Gordan coefficients* for $SO(3)$ are then defined as $\{C_{l_1 m_1 l_2 m_2}^{lm}\}$, that is, as the elements of the unitary matrices $C_{l_1 l_2}$ (note that such matrices are real-valued, and so are the

$C_{l_1 m_1 l_2 m_2}^{lm}$). These coefficients were introduced in Mathematics in the XIX century, as motivated by the analysis of invariants in Algebraic Geometry; in the 20th century, they have gained an enormous importance in the quantum theory of angular momentum, where $C_{l_1 m_1 l_2 m_2}^{lm}$ represents the *probability amplitude* that two quantum particles with total angular momentum l_1 and l_2 and momentum projections on the z -axis m_1 and m_2 are coupled to form a system with total angular momentum l and projection m (see e.g. [13]). Their use in the analysis of isotropic random fields is much more recent, see for instance [9] and the references therein. Explicit expressions for the Clebsch-Gordan coefficients of $SO(3)$ are known, but they are in general hardly manageable (see e.g. [28, Section 8.2]). However, these expressions become somewhat neater when $m_1 = m_2 = m_3 = 0$, in which case one has the relations: $C_{l_1 0 l_2 0}^{l_3 0} = 0$, when $l_1 + l_2 + l_3$ is odd, and, for $l_1 + l_2 + l_3$ even,

$$C_{l_1 0 l_2 0}^{l_3 0} = \frac{(-1)^{\frac{l_1 + l_2 - l_3}{2}} \sqrt{2l_3 + 1} [(l_1 + l_2 + l_3)/2]!}{[(l_1 + l_2 - l_3)/2]! [(l_1 - l_2 + l_3)/2]! [(-l_1 + l_2 + l_3)/2]!} \times \left\{ \frac{(l_1 + l_2 - l_3)!(l_1 - l_2 + l_3)!(-l_1 + l_2 + l_3)!}{(l_1 + l_2 + l_3 + 1)!} \right\}^{1/2}.$$

The Clebsch-Gordan coefficients enjoy also a nice set of symmetry and orthogonality properties which will play a crucial role in our results to follow (see [14] and [15] for an account of such properties). Note in particular that the Clebsch-Gordan coefficients are different from zero only if $m_1 + m_2 = m$ and $|l_2 - l_1| \leq l \leq l_1 + l_2$ (the *triangle conditions*). Also, from unitary equivalence we deduce that

$$\sum_{m_1, m_2} C_{l_1 m_1 l_2 m_2}^{lm} C_{l_1 m'_1 l_2 m'_2}^{l'm'} = \delta_l^{l'} \delta_m^{m'} \quad \text{and} \quad \sum_{l, m} C_{l_1 m_1 l_2 m_2}^{lm} C_{l_1 m'_1 l_2 m'_2}^{lm} = \delta_{m_1}^{m'_1} \delta_{m_2}^{m'_2}. \quad (28)$$

Remark on Notation. Depending on the notational convenience, we write sometimes sums of Clebsch-Gordan coefficients without specifying the range of the indices l and/or m . In such cases, the range of the sums is conventionally taken to be the set of indices where the Clebsch-Gordan coefficients are different from zero. For instance, in (28) one should read: $\sum_{m_1, m_2} = \sum_{m_1=-l_1, \dots, l_1} \sum_{m_2=-l_2, \dots, l_2}$ and $\sum_{l, m} = \sum_{l=0}^{+\infty} \sum_{m=-l, \dots, l}$. Similar conventions are adopted (without further notice) throughout the paper. We recall also that the Clebsch-Gordan coefficients are equivalent, up to a normalization factor, to the *Wigner's 3j coefficients*, which are used in related works such as [15].

The Clebsch-Gordan coefficients play a crucial role in the evaluation of integrals involving products of spherical harmonics. In particular, the so-called *Gaunt integral* gives

$$\int_{\mathbb{S}^2} Y_{l_1 m_1}(x) Y_{l_2 m_2}(x) \overline{Y_{lm}(x)} dx = \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} C_{l_1 m_1 l_2 m_2}^{lm} C_{l_1 0 l_2 0}^{l0}. \quad (29)$$

Relation (29) can be established using (26), (27) and resorting to standard orthonormality properties of the elements of group representations – see [28, Expression 5.9.1.4]. More generally, define

$$\mathcal{G}\{l_1, m_1; \dots; l_r, m_r\} \triangleq \int_{\mathbb{S}^2} Y_{l_1 m_1}(x) \cdots Y_{l_r m_r}(x) dx, \quad (30)$$

and call the quantity $\mathcal{G}\{l_1, m_1; \dots; l_r, m_r\}$ a *generalized Gaunt integral*. Then, iterating the previous argument, for $q \geq 3$ it can be shown that (by using for instance [28, Expression

5.6.2.12])

$$\begin{aligned}
& \mathcal{G} \{l_1, m_1; \dots; l_q, m_q; l, -m\} \\
&= \sum_{L_1 \dots L_{q-2}} \sum_{M_1 \dots M_{q-2}} \left\{ \prod_{i=1}^{q-3} \left(\sqrt{\frac{2l_{i+2}+1}{4\pi}} C_{L_i 0 l_{i+2} 0}^{L_{i+1} 0} C_{L_i M_i l_{i+2} m_{i+2}}^{L_{i+1} M_{i+1}} \right) \right\} \\
& \times \sqrt{\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}} C_{l_1 0 l_2 0}^{L_1 0} C_{l_1 m_1 l_2 m_2}^{L_1 M_1} \sqrt{\frac{2l_q+1}{4\pi}} C_{L_{q-2} 0 l_q 0}^{l 0} C_{L_{q-2} M_{q-2} l_q m_q}^{l m},
\end{aligned} \tag{31}$$

where, for $q = 3$, we have used the convention $\prod_{i=1}^0 \equiv 0$. Note that expressions such as (31) imply that the generalized Gaunt integrals of the type (30) are indeed real-valued. To simplify the expression (31), let us introduce the coefficients

$$C_{l_1, m_1; \dots; l_p, m_p}^{\lambda_1, \lambda_2, \dots, \lambda_{p-1}; \mu} \triangleq \sum_{\mu_1 = -\lambda_1}^{\lambda_1} \dots \sum_{\mu_{p-2} = -\lambda_{p-2}}^{\lambda_{p-2}} C_{l_1, m_1, l_2, m_2}^{\lambda_1, \mu_1} C_{\lambda_1, \mu_1; l_3, m_3}^{\lambda_2, \mu_2} \dots C_{\lambda_{p-2}, \mu_{p-2}; l_p, m_p}^{\lambda_{p-1}, \mu}.$$

These coefficients are themselves the elements of unitary matrices connecting tensor product and direct sum representations of $SO(3)$, and thus it follows easily that the following orthonormality conditions hold

$$\sum_{m_1, \dots, m_p} \left\{ C_{l_1, m_1; \dots; l_p, m_p}^{\lambda_1, \lambda_2, \dots, \lambda_{p-1}; \mu} \right\}^2 = \sum_{\lambda_1} \dots \sum_{\lambda_{p-1}} \sum_{\mu = -\lambda_{p-1}}^{\lambda_{p-1}} \left\{ C_{l_1, m_1; \dots; l_p, m_p}^{\lambda_1, \lambda_2, \dots, \lambda_{p-1}; \mu} \right\}^2 = 1; \tag{32}$$

it is important to note that due to the conditions $m_1 + m_2 = m_3$ the sums may actually vanish, for instance

$$C_{l_1, 0; \dots; l_p, 0}^{\lambda_1, \lambda_2, \dots, \lambda_{p-1}; 0} = C_{l_1, 0, l_2, 0}^{\lambda_1, 0} C_{\lambda_1, 0; l_3, 0}^{\lambda_2, 0} \dots C_{\lambda_{p-2}, 0; l_p, 0}^{\lambda_{p-1}, 0}. \tag{33}$$

We have also that

$$\begin{aligned}
& \mathcal{G} \{l_1, m_1; \dots; l_q, m_q; l, -m\} \\
&= \sqrt{\frac{4\pi}{2l+1}} \left\{ \prod_{i=1}^q \sqrt{\frac{2l_i+1}{4\pi}} \right\} \sum_{L_1 \dots L_{q-2}} C_{l_1, 0; \dots; l_q, 0}^{L_1, L_2, \dots, L_{q-2}, l; 0} C_{l_1, m_1; \dots; l_q, m_q}^{L_1, L_2, \dots, L_{q-2}, l; m}.
\end{aligned} \tag{34}$$

Remark. The coefficients $C_{l_1, m_1; \dots; l_p, m_p}^{\lambda_1, \lambda_2, \dots, \lambda_{p-1}; \mu}$ defined above admit a physical interpretation in terms of coupling of angular momenta in a quantum mechanical system. Consider indeed a system composed of p particles, say $\alpha_1, \dots, \alpha_p$, such that α_i has total angular momentum equal to l_i , and projection on the z -axis given by m_i . Then, the coefficient $C_{l_1, m_1; \dots; l_p, m_p}^{\lambda_1, \lambda_2, \dots, \lambda_{p-1}; \mu}$ is exactly the probability amplitude of the intersection of the following $p-1$ events $\mathbf{E}_1, \dots, \mathbf{E}_{p-1}$: $\mathbf{E}_1 = \{\alpha_1 \text{ and } \alpha_2 \text{ couple to form a particle } \eta_1 \text{ with total angular momentum } \lambda_1\}$, $\mathbf{E}_2 = \{\eta_1 \text{ couples with } \alpha_3 \text{ to form a particle } \eta_2 \text{ with total angular momentum } \lambda_2\}$, \dots , $\mathbf{E}_i = \{\eta_{i-1} \text{ couples with } \alpha_{i+1} \text{ to form a particle } \eta_i \text{ with total angular momentum } \lambda_i\}$, \dots , $\mathbf{E}_{p-1} = \{\eta_{p-2} \text{ couples with } \alpha_p \text{ to form a particle with total angular momentum } \lambda_{p-1} \text{ and projection } \mu \text{ on the } z\text{-axis}\}$.

In the sequel, we shall also need the so-called *Wigner 6j (or Racah) coefficients*, which are related to the Clebsch-Gordan by the identity (see ([28, Eq. 9.1.1.8]))

$$\left\{ \begin{matrix} l_1 & l_2 & l_3 \\ l_4 & l_5 & l_6 \end{matrix} \right\} = K(l_1, \dots, l_6) \sum_{\substack{m_1 m_3 \\ m_4 m_6}} C_{l_1 m_1 l_2 m_2}^{l_3 m_3} C_{l_1 m_1 l_6 m_6}^{l_5 m_5} C_{l_3 m_3 l_4 m_4}^{l_5 m_5} C_{l_2 m_2 l_4 m_4}^{l_6 m_6}. \tag{35}$$

where $K(l_1, \dots, l_6) = [(2l_3 + 1)(2l_6 + 1)]^{-1/2} (-1)^{l_1 + l_2 + l_4 + l_5}$ (note that the previous sum does not involve m_2 and m_5 , because of the general relation: $C_{\alpha t_1 \beta t_2}^{\gamma t_3} = 0$, whenever $t_3 \neq t_1 + t_2$). Although the Wigner's 6j coefficients play themselves a very important role in Quantum Mechanics and Representation Theory, for brevity's sake we avoid a full discussion on their properties; the interested reader can consult ([28, Ch.9]) or ([29, pp. 529-542]).

4 High-frequency CLTs: conditions in terms of Gaunt integrals

The aim of this section is to obtain conditions for high-frequency CLTs in terms of Gaunt integrals of the type (31). We start by focussing on Hermite polynomials, and then we deal with general subordinated fields.

4.1 Hermite subordination

We focus on the spherical field $T^{(q)}$ ($q \geq 2$) defined in (15), which is obtained by composing the Gaussian field T in (2) with the q th Hermite polynomial H_q (or, equivalently, by taking the q th Wick power of the random variable $T(x)$ for every x). Our first purpose is to characterize the asymptotic Gaussianity (when $l \rightarrow +\infty$) of the spherical harmonic coefficients $\{a_{lm;q}\}$ defined in (16).

Theorem 3 Fix $q \geq 2$.

1. For every $l \geq 1$, the positive constant $\tilde{C}_l^{(q)}$ in (19) (which does not depend on m) equals the quantity

$$q! \sum_{l_1, m_1} \dots \sum_{l_q, m_q} C_{l_1} C_{l_2} \dots C_{l_q} |\mathcal{G}\{l_1, m_1; \dots; l_q, m_q; l, -m\}|^2 \quad (36)$$

$$= q! \sum_{l_1, \dots, l_q=0}^{\infty} C_{l_1} \dots C_{l_q} \frac{4\pi}{2l+1} \left\{ \prod_{i=1}^q \frac{2l_i + 1}{4\pi} \right\} \sum_{L_1 \dots L_{q-2}} \left\{ C_{l_1, 0; \dots; l_q, 0}^{L_1, L_2, \dots, L_{q-2}, l; 0} \right\}^2 \quad (37)$$

for every $m = -l, \dots, l$, where the (generalized) Gaunt integral $\mathcal{G}\{\cdot\}$ is defined via (30).

2. Fix $m \neq 0$. As $l \rightarrow +\infty$, the following two conditions **(A)** and **(B)** are equivalent: **(A)**

$$(\tilde{C}_l^{(q)})^{-1/2} \times a_{lm;q} \xrightarrow{\text{law}} N + iN', \quad (38)$$

where $N, N' \sim \mathcal{N}(0, 1/2)$ are independent; **(B)** for every $p = \frac{q-1}{2} + 1, \dots, q-1$, if $q-1$ is even, and every $p = q/2, \dots, q-1$ if $q-1$ is odd

$$\begin{aligned} & (\tilde{C}_l^{(q)})^{-2} \sum_{n_1, j_1} \dots \sum_{n_{2(q-p)}, j_{2(q-p)}} C_{j_1} \dots C_{j_{2(q-p)}} \left| \sum_{l_1, m_1} \dots \sum_{l_p, m_p} C_{l_1} \dots C_{l_p} \right. \\ & \times \mathcal{G}\{l_1, m_1; \dots; l_p, m_p; j_1, n_1; \dots; j_{q-p}, n_{q-p}; l, -m\} \times \\ & \left. \times \mathcal{G}\{l_1, m_1; \dots; l_p, m_p; j_{q-p+1}, n_{q-p+1}; \dots; j_{2(q-p)}, n_{2(q-p)}; l, -m\} \right|^2 \rightarrow 0 \end{aligned} \quad (39)$$

3. Let N be a centered Gaussian random variable with unitary variance. As $l \rightarrow +\infty$, the CLT

$$(\tilde{C}_l^{(q)})^{-1/2} \times a_{l0;q} \xrightarrow{\text{law}} N \quad (40)$$

takes place if, and only if, the asymptotic condition (39) holds for $m = 0$ and for every $p = \frac{q-1}{2} + 1, \dots, q-1$, if $q-1$ is even, and every $p = q/2, \dots, q-1$ if $q-1$ is odd.

Proof. Consider a standard Brownian motion $W = \{W_t : t \in [0, 1]\}$, and denote by $L_{\mathbb{C}}^2([0, 1]) = L_{\mathbb{C}}^2([0, 1], d\lambda)$ the class of complex-valued and square integrable functions on $[0, 1]$, with respect to the Lebesgue measure $d\lambda$. Now select a complex-valued family $\{g_{lm} : l \geq 0, -l \leq m \leq l\} \subseteq L_{\mathbb{C}}^2([0, 1])$ with the following five properties: (1) g_{l0} is real for every $l \geq 0$, (2) $g_{lm} = (-1)^m \overline{g_{l-m}}$, (3) $\int g_{lm} \overline{g_{l'm'}} d\lambda = 0, \forall (l, m) \neq (l', m')$, (4) $\int \Re(g_{lm}) \Im(g_{lm}) d\lambda = 0$, (5) $\int \Re(g_{lm})^2 d\lambda = \int \Im(g_{lm})^2 d\lambda = \int g_{l0}^2 d\lambda / 2 = C_l / 2$, where $\{C_l : l \geq 0\}$ is the power spectrum of the Gaussian field T . According to Proposition 1, the following identity in law holds:

$$\{a_{lm;1} : l \geq 0, -l \leq m \leq l\} \stackrel{\text{law}}{=} \{I_1(g_{lm}) : l \geq 0, -l \leq m \leq l\},$$

where $I_1(g_{lm}) = \int_0^1 g_{lm} dW = \int_0^1 \Re(g_{lm}) dW + i \int_0^1 \Im(g_{lm}) dW$ is the usual (complex-valued) Wiener-Itô integral of g_{lm} with respect to W . From this last relation, it also follows that, in the sense of stochastic processes, $T(x) \stackrel{\text{law}}{=} I_1\left(\sum_{l=0}^{\infty} \sum_{m=-l}^l g_{lm} Y_{lm}(x)\right)$ (note that the function $z \mapsto \sum_{l,m} g_{lm}(z) Y_{lm}(x)$ is real-valued for every fixed $x \in \mathbb{S}^2$ and with norm equal to 1). Now define $L_{s,\mathbb{C}}^2([0, 1]^q)$ to be the class of complex-valued and symmetric functions on $[0, 1]^q$, that are square-integrable with respect to Lebesgue measure. For every $f \in L_{s,\mathbb{C}}^2([0, 1]^q)$, we define $I_q(f) = I_q(\Re(f)) + i I_q(\Im(f))$ to be the multiple Wiener-Itô integral, of order q , of f with respect to the Brownian motion W (see e.g. [18, Ch. 1], or [11]). From the previous discussion it follows that, for every $q \geq 2$,

$$T^{(q)}(x) = H_q(T(x)) \stackrel{\text{law}}{=} I_q \left[\left\{ \sum_{l=0}^{\infty} \sum_{m=-l}^l g_{lm} Y_{lm}(x) \right\}^{\otimes q} \right], \quad (41)$$

where the equality in law holds in the sense of finite dimensional distributions and, for every $f \in L_{\mathbb{C}}^2([0, 1])$, we use the notation $f^{\otimes q}(a_1, \dots, a_q) = f(a_1) \times \dots \times f(a_q)$. Note that, to obtain the last equality in (41), we used the well-known relation (see e.g. [11]): for every real-valued $f \in L_{\mathbb{R}}^2([0, 1])$ such that $\|f\|_{L_{\mathbb{R}}^2([0, 1])} = 1$, it holds that $H_q[I_1(f)] = I_q(f^{\otimes q})$. Now set $h_{l,m}^{(q)} = (-1)^m \sum_{l_1, m_1} \dots \sum_{l_q, m_q} g_{l_1 m_1} \dots g_{l_q m_q} \mathcal{G}\{l_1, m_1; \dots; l_q, m_q; l, -m\}$, so that

$$a_{lm;q} \stackrel{\text{law}}{=} \int_{S^2} I_q \left[\left\{ \sum_{l=0}^{\infty} \sum_{m=-l}^l g_{lm} Y_{lm}(x) \right\}^{\otimes q} \right] \overline{Y_{lm}(x)} dx = I_q[h_{l,m}^{(q)}] \quad (42)$$

so that (36) follows immediately from the well-known isometry relation:

$$\mathbb{E} \left[\left| I_q[h_{l,m}^{(q)}] \right|^2 \right] = q! \|h_{l,m}^{(q)}\|_{L^2([0, 1]^q)}^2$$

(to obtain (42) we interchanged stochastic and deterministic integration, by means of a standard stochastic Fubini argument). To prove that (37) is equal to (36), observe first that (32) yields that

$$\sum_{m_1=-l_1}^{l_1} \dots \sum_{m_q=-l_q}^{l_q} C_{l_1, m_1; \dots; l_q, m_q}^{L_1, L_2, \dots, L_{q-2}, l; m} C_{l_1, m_1; \dots; l_q, m_q}^{L'_1, L'_2, \dots, L'_{q-2}, l; m} = \delta_{L_1}^{L'_1} \dots \delta_{L_{q-2}}^{L'_{q-2}}$$

(the RHS of the previous expression does not depend on m). Then, use (34) to deduce that

$$\begin{aligned} & \sum_{m_1=-l_1}^{l_1} \cdots \sum_{m_q=-l_q}^{l_q} \mathcal{G}\{l_1, m_1; \dots; l_q, m_q; l, -m\}^2 \\ &= \frac{4\pi}{2l+1} \left\{ \prod_{i=1}^q \frac{2l_i+1}{4\pi} \right\} \sum_{L_1 \dots L_{q-2}} \left\{ C_{l_1,0;\dots;l_{q-2},0}^{L_1,L_2,\dots,L_{q-2},l;0} \right\}^2. \end{aligned}$$

This proves Point 1 in the statement. To prove Point 2, recall that, according to [17, Proposition 6], relation (38) holds if, and only if,

$$(\tilde{C}_l^{(q)})^{-2} \left\| h_{l,m}^{(q)} \otimes_p \overline{h_{l,m}^{(q)}} \right\|_{L^2([0,1]^{2(q-p)})}^2 \rightarrow 0,$$

for every $p = 1, \dots, q-1$, where the complex-valued (and not necessarily symmetric) function $h_{l,m}^{(q)} \otimes_p \overline{h_{l,m}^{(q)}}$ (which is an element of $L^2([0,1]^{2(q-p)})$) is defined as the *contraction*

$$\begin{aligned} & h_{l,m}^{(q)} \otimes_p \overline{h_{l,m}^{(q)}}(a_1, \dots, a_{2(q-p)}) \\ &= \int_{[0,1]^p} h_{l,m}^{(q)}(\mathbf{x}_p, a_1, \dots, a_{q-p}) \overline{h_{l,m}^{(q)}(\mathbf{x}_p, a_{q-p+1}, \dots, a_{2(q-p)})} d\mathbf{x}_p, \end{aligned} \tag{43}$$

for every $(a_1, \dots, a_{2(q-p)}) \in [0,1]^{2(q-p)}$, where $d\mathbf{x}_p$ is the Lebesgue measure on $[0,1]^p$. Since, trivially, $\|h_{l,m}^{(q)} \otimes_p \overline{h_{l,m}^{(q)}}\|^2 = \|h_{l,m}^{(q)} \otimes_{q-p} \overline{h_{l,m}^{(q)}}\|^2$ (we stress that, in the last equality, the first norm is taken in $L^2([0,1]^{2(q-p)})$, whereas the second is in $L^2([0,1]^{2p})$), one deduces that it is sufficient to check that the norm of $h_{l,m}^{(q)} \otimes_p \overline{h_{l,m}^{(q)}}$ is asymptotically negligible for every $p = \frac{q-1}{2} + 1, \dots, q-1$, if $q-1$ is even, and every $p = q/2, \dots, q-1$ if $q-1$ is odd. It follows that the result is proved once it is shown that, for every p in such range, the norm $\|h_{l,m}^{(q)} \otimes_p \overline{h_{l,m}^{(q)}}\|^2$ equals the multiple sum appearing in (39). To see this, use (43) to deduce that (recall that Gaunt integrals are real-valued)

$$\begin{aligned} & h_{l,m}^{(q)} \otimes_p \overline{h_{l,m}^{(q)}}(a_1, \dots, a_{2(q-p)}) \\ &= \sum_{n_1, j_1} \cdots \sum_{n_{2(q-p)}, j_{2(q-p)}} g_{j_1 n_1} \cdots g_{j_{q-p} n_{q-p}} \overline{g_{j_{q-p+1} n_{q-p+1}} \cdots g_{j_{2(q-p)} n_{2(q-p)}}} \\ & \quad \sum_{l_1, m_1} \cdots \sum_{l_p, m_p} C_{l_1} \cdots C_{l_p} \mathcal{G}\{l_1, m_1; \dots; l_p, m_p; j_1, n_1; \dots; j_{q-p}, n_{q-p}; l, -m\} \\ & \quad \mathcal{G}\{l_1, m_1; \dots; l_p, m_p; j_{q-p+1}, n_{q-p+1}; \dots; j_{2(q-p)}, n_{2(q-p)}; l, -m\}, \end{aligned}$$

and the result is obtained by using the orthogonality properties of the g_{jn} 's. Point 3 in the statement is proved in exactly the same way, by first observing that $a_{l0;q}$ is a real-valued random variable, and then by applying Theorem 1 in [19]. ■

Remark. One has the relation $\mathbb{E} \left[T^{(q)}(x)^2 \right] = q! [E \{T(x)^2\}]^q$. This equality can be proved in two ways: (i) by exploiting the representation of $T^{(q)}(x)$ as a multiple Wiener-Itô integral,

or (ii) by using the equality $\mathbb{E} \left[T^{(q)}(x)^2 \right] = \sum_l \frac{2l+1}{4\pi} \tilde{C}_l^{(q)}$, and the by expanding $\tilde{C}_l^{(q)}$ according to Theorem 3, so that one can apply the orthogonality relations (32).

Now recall that, according to part 2 of Lemma 2, $T_l^{(q)}(x) \stackrel{\text{law}}{=} \sqrt{\frac{2l+1}{4\pi}} a_{l0;q}$, so that relation (22) holds. This gives immediately a first (exhaustive) solution to Problem **(P-I)**, as stated in Section 2.

Corollary 4 *For every $q \geq 2$ the following conditions are equivalent:*

1. *The CLT (23) holds for every $x \in \mathbb{S}^2$;*
2. *The asymptotic relation (39) takes place for $m = 0$ and for every $p = \frac{q-1}{2} + 1, \dots, q-1$, if $q-1$ is even, and every $p = q/2, \dots, q-1$ if $q-1$ is odd.*

To deal with Problem **(P-II)** of Section 2, we recall the notation $\overline{T}_l^{(q)}$ (indicating the l th normalized frequency component of $T^{(q)}$) introduced in (18). We also introduce (for every $l \geq 1$) the *normalized l th frequency component* of the Gaussian field T , which is defined as

$$\overline{T}_l(x) = \frac{T_l(x)}{\text{Var}(T_l(x))^{1/2}} = \frac{T_l(x)}{(\frac{2l+1}{4\pi} C_l)^{1/2}}, \quad x \in \mathbb{S}^2. \quad (44)$$

According to Lemma 2 (in the special case $F(z) = z$), \overline{T}_l is a real-valued, isotropic, centered and Gaussian field. Moreover, one has that $\mathbb{E}[\overline{T}_l(x) \overline{T}_l(y)] = \mathbb{E}[\overline{T}_l^{(q)}(x) \overline{T}_l^{(q)}(y)] = P_l(\langle x, y \rangle)$, for every $q \geq 2$ and every $l \geq 1$. The next result – which gives an exhaustive solution to Problem **(P-II)** – states that, whenever Condition 1 (or, equivalently, Condition 2) in the statement of Corollary 4 is verified (and without any additional assumption), the “distance” between the finite dimensional distributions of the normalized field $\overline{T}_l^{(q)}$ and those of \overline{T}_l converge to zero. For every $k \geq 1$, we denote by $\mathbf{P}(\mathbb{R}^k)$ the class of all probability measures on \mathbb{R}^k . We say that a metric $\gamma(\cdot, \cdot)$ *metrizes the weak convergence on $\mathbf{P}(\mathbb{R}^k)$* whenever the following double implication holds for every $Q \in \mathbf{P}(\mathbb{R}^k)$ and every $\{Q_l : l \geq 1\} \subset \mathbf{P}(\mathbb{R}^k)$ (as $l \rightarrow +\infty$): $\gamma(Q_l, Q) \rightarrow 0$ if, and only if, Q_l converges weakly to Q . The quantity $\gamma(P, Q)$ is sometimes called the γ -distance between P and Q .

Theorem 5 *Let $q \geq 2$ be fixed, and suppose that Condition 1 (or 2) of Corollary 4 is satisfied.*

1. *For every $k \geq 1$, every $x_1, \dots, x_k \in \mathbb{S}^2$ and every compact subset $M \subset \mathbb{R}^k$,*

$$\sup_{(\lambda_1, \dots, \lambda_k) \in M} \left| \mathbb{E} \left[e^{i \sum_{j=1}^k \lambda_j \overline{T}_l^{(q)}(x_j)} \right] - \mathbb{E} \left[e^{i \sum_{j=1}^k \lambda_j \overline{T}_l(x_j)} \right] \right| \xrightarrow{l \rightarrow +\infty} 0. \quad (45)$$

2. *Fix x_1, \dots, x_k and denote by $\mathcal{L}(\overline{T}_l^{(q)}; x_1, \dots, x_k)$ and $\mathcal{L}(\overline{T}_l; x_1, \dots, x_k)$ ($l \geq 1$), respectively, the law of $(\overline{T}_l^{(q)}(x_1), \dots, \overline{T}_l^{(q)}(x_k))$ and the law of $(\overline{T}_l(x_1), \dots, \overline{T}_l(x_k))$. For every metric $\gamma(\cdot, \cdot)$ on $\mathbf{P}(\mathbb{R}^k)$ such that $\gamma(\cdot, \cdot)$ metrizes the weak convergence, it holds that*

$$\lim_{l \rightarrow +\infty} \gamma \left(\mathcal{L}(\overline{T}_l^{(q)}; x_1, \dots, x_k), \mathcal{L}(\overline{T}_l; x_1, \dots, x_k) \right) = 0.$$

Proof. The crucial point is that the spherical field $x \mapsto \overline{T}_l^{(q)}(x)$ lives in the q th Wiener chaos associated with the Gaussian space generated by T . By using this fact, and by arguing as in the proof of Theorem 3, one can show that the vector $(\overline{T}_l^{(q)}(x_1), \dots, \overline{T}_l^{(q)}(x_k))$ is indeed equal in law to a vector of multiple Wiener-Itô integrals, of order q , with respect to a Brownian motion. Since each element of this vector converges in law to a standard Gaussian random variable, one can directly apply Theorem 1 and Proposition 2 in [21] to achieve the desired conclusion (see also [21, Proposition 5]). ■

4.2 General subordination

We now give a solution to Problem (P-III), as stated at the end of Section 2, where F is a general real-valued function belonging to the class $L_0^2(\mathbb{R}, e^{-x^2/2} dx)$. The function F admits a unique representation of the form

$$F(z) = \sum_{q=1}^{\infty} \frac{c_q(F)}{q!} H_q(z), \quad z \in \mathbb{R}, \quad (46)$$

where the Hermite polynomials H_q are given by (14) and the real coefficients $c_q(F)$, $q = 1, 2, \dots$, are such that

$$\sum_q \frac{c_q(F)^2}{q!} < +\infty. \quad (47)$$

As a consequence, for every $l \geq 0$, the frequency component $F[T]_l(x)$ defined in (13) admits the expansion

$$F[T]_l(x) = \sum_{q=1}^{\infty} \frac{c_q(F)}{q!} T_l^{(q)}(x), \quad x \in \mathbb{S}^2, \quad (48)$$

where the series converges in $L^2(\mathbb{P})$ for every fixed x . Formula (48) combined with Lemma 2 yields also that

$$\mathbb{E}(F[T]_l(x) F[T]_l(y)) = \frac{2l+1}{4\pi} P_l(\cos \langle x, y \rangle) \sum_{q=1}^{\infty} \left(\frac{c_q(F)}{q!} \right)^2 \tilde{C}_l^{(q)},$$

where $\tilde{C}_l^{(q)}$ is given by (19) or, equivalently, by (37). The next result characterizes the asymptotic Gaussianity of F -subordinated spherical random fields. Recall the definition of $\overline{F}[T]_l$ given in (20). The proof is standard, and therefore omitted (it can be obtained e.g. along the lines of [10, Th. 4]).

Theorem 6 *Suppose that the following relations hold*

1. *For every $q \geq 1$, $\lim_{l \rightarrow +\infty} \frac{2l+1}{4\pi} \left(\frac{c_q(F)}{q!} \right)^2 \tilde{C}_l^{(q)} / \mathbb{E}(F[T]_l(x)^2) \rightarrow \sigma_q^2 \in [0, +\infty)$;*
2. $\sum_{m \geq 1} \{c_q(F)/q!\}^2 \sigma_q^2 \triangleq \sigma^2(F) < +\infty$;
3. *For every $q \geq 2$, the asymptotic relation (39) takes place for $m = 0$ and for every $p = \frac{q-1}{2} + 1, \dots, q-1$, if $q-1$ is even, and every $p = q/2, \dots, q-1$ if $q-1$ is odd;*

$$4. \lim_{p \rightarrow +\infty} \overline{\lim}_l (2l+1) \sum_{q=p+1}^{\infty} \left(\frac{c_q(F)}{q!} \right)^2 \tilde{C}_l^{(q)} = 0.$$

Then, for every $k \geq 1$, every $x_1, \dots, x_k \in \mathbb{S}^2$ and every compact $M \subset \mathbb{R}^k$,

$$\sup_{(\lambda_1, \dots, \lambda_k) \in M} \left| \mathbb{E} \left[e^{i \sum_{j=1}^k \lambda_j \overline{F[T]_l}(x_j)} \right] - \mathbb{E} \left[e^{i \sigma^2(F)^{1/2} \sum_{j=1}^k \lambda_j \overline{T}_l(x_j)} \right] \right| \xrightarrow{l \rightarrow +\infty} 0,$$

where we used the notation (44). In particular, the last asymptotic relation implies that, for every $\gamma(\cdot, \cdot)$ metrizing the weak convergence on $\mathbf{P}(\mathbb{R}^k)$, the γ -distance between

$$(\overline{F[T]_l}(x_1), \dots, \overline{F[T]_l}(x_k))$$

and $\sigma^2(F)^{1/2}(\overline{T}_l(x_1), \dots, \overline{T}_l(x_k))$ converges to zero as $l \rightarrow +\infty$.

Remark. A sufficient condition, ensuring that points 1 and 3 in the statement of Theorem 6 are verified, is the following: there exist constants $\rho(q) > 0$ such that (a) $(2l+1) \tilde{C}_l^{(q)} \leq \rho(q)$ for every $q \geq 1$ and every l , and (b) $\sum_{q=1}^{\infty} \left(\frac{c_q(F)}{q!} \right)^2 \rho(q) < +\infty$.

5 Explicit sufficient conditions: convolutions and random walks

In this section, we further explicit the conditions for the CLTs proved in Section 4 for the (Hermite) frequency components $T_l^{(q)}$, $l \geq 0$. In particular, we shall establish sufficient conditions that are more directly linked to primitive assumptions on the behaviour of the angular power spectrum $\{C_l : l \geq 0\}$. The results of Section 5.2 and Section 5.3 cover, respectively, the case $q = 2$ and $q = 3$. Section 5.4 contains some partial findings for the case of a general q , as well as several conjectures. These results will be used in Section 7 to deduce explicit conditions on the rate of decay of the angular power spectrum $\{C_l : l \geq 0\}$.

Our analysis is inspired by the following result, which is a particular case of the statements contained in [17, Section 3], concerning fields on Abelian groups. Consider indeed a centered real-valued Gaussian field $V = \{V(\theta) : \theta \in \mathbb{T}\}$ defined on the torus $\mathbb{T} = [0, 2\pi)$ (that we regard as an Abelian compact group with group operation given by $xy = (x+y) \bmod(2\pi)$). We suppose that the law of V is *isotropic*, i.e. that $V(\theta) \stackrel{\text{law}}{=} V(x\theta)$ (in the sense of stochastic processes) for every $x \in \mathbb{T}$, and also $\mathbb{E} V(\theta)^2 = 1$. We denote by $V(\theta) = \sum_{l \in \mathbb{Z}} a_l e^{il\theta}$ the Fourier decomposition of V , and we write $\Gamma_l^V = \mathbb{E} |a_l|^2$ (note that $\Gamma_l^V = \Gamma_{-l}^V$). Fix $q \geq 2$, and consider the Hermite-subordinated field $H_q[V](\theta) = H_q(V(\theta))$, where q is the q th Hermite polynomial. The Fourier decomposition of $H_q[V]$ is $H_q[V](\theta) = \sum_{l \in \mathbb{Z}} a_l^{(q)} e^{il\theta}$. We write N, N' to indicate a pair of independent centered Gaussian random variables with common variance equal to $1/2$: in [17] it is proved that to have the *high-frequency CLT*

$$\frac{a_l^{(q)}}{\text{Var}(a_l^{(q)})^{1/2}} = \frac{\int_{\mathbb{T}} H_q[V](\theta) e^{-il\theta} d\theta}{\text{Var}(a_l^{(q)})^{1/2}} \xrightarrow[l \rightarrow \infty]{\text{law}} N + iN' \quad (49)$$

it is *necessary and sufficient* that, for every $p = 1, \dots, q-1$,

$$\lim_{l \rightarrow +\infty} \sup_{j \in \mathbb{Z}} \mathbb{P}[U_p = j \mid U_q = l] = 0, \quad (50)$$

where $\{U_n : n \geq 0\}$ is the random walk on \mathbb{Z} whose law is given by $U_0 = 0$ and

$$\mathbb{P}[U_{n+1} = j \mid U_n = k] = \Gamma_{j-k}^V.$$

Note that the law of the random variable U_n has trivially the form of a *convolution* of the coefficients Γ_l^V (see also the discussion below). The correspondence between (49) and the “random walk bridge” (50) has been used in [17] to establish explicit conditions on the power spectrum $\{\Gamma_l^V\}$ to have that (49) holds.

In what follows, we shall unveil (and apply) an analogous connection between the CLTs proved in Section 4 and some specific convolutions and random walks on $\widehat{SO(3)}$.

5.1 Convolutions on $\widehat{SO(3)}$

In the light of Part 3 of Theorem 3 and by Corollary 4, we will focus on the sequence $\{a_{l0;q} : l \geq 0\}$ (see (16)), whose behaviour as $l \rightarrow +\infty$ yields an asymptotic characterization of the fields $T_l^{(q)}(\cdot)$ defined in (17). A crucial point is the simple fact that the numerator of (39), for $m = 0$, can be developed as a multiple sum involving products of four generalized Gaunt integrals, so that, by (31), the asymptotic expressions appearing in Theorem 3 can be studied by means of the properties of linear combinations of products of Clebsch-Gordan coefficients. As anticipated, a very efficient tool for our analysis will be the use of convolutions on \mathbb{N} , that we endow with an hypergroup structure isomorphic to $\widehat{SO(3)}$, i.e. the dual of $SO(3)$. This will be the object of the subsequent discussion.

From now on, and for the rest of the section, we shall fix a sequence $\{C_l : l \geq 0\}$, representing the angular power spectrum of an isotropic centered, normalized Gaussian field T over \mathbb{S}^2 , as in Section 2. Whenever convenient we shall write

$$\Gamma_l \triangleq (2l+1)C_l, \quad l \geq 0, \quad (51)$$

so that, for $l \geq 1$ and up to the constant $1/4\pi$, the parameter Γ_l represents the variance of the projection of the Gaussian field T in (2) on the frequency l : indeed, according to Lemma 2, $\text{Var}(T_l) = \Gamma_l/4\pi$. Also, we define the following *convolutions* of the coefficients Γ_l (in the following expressions, the sums over indices $l_i, L_i \dots$ range implicitly from 0 to $+\infty$):

$$\widehat{\Gamma}_{2,l} = \sum_{l_1, l_2} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l 0})^2, \quad (52)$$

$$\widehat{\Gamma}_{3,l} = \sum_{L_1, L_2} \widehat{\Gamma}_{2, L_1} \Gamma_{L_2} (C_{L_1 0 L_2 0}^{l 0})^2 = \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \sum_{L_1} (C_{l_1 0 l_2 0 L_1 0}^{L_1 l; 0})^2, \dots \quad (53)$$

$$\widehat{\Gamma}_{q,l} = \sum_{L_1, L_q} \widehat{\Gamma}_{q-1, L_{q-1}} \Gamma_{L_q} (C_{L_{q-1} 0 L_q 0}^{l 0})^2 = \sum_{l_1 \dots l_q} \Gamma_{l_1} \dots \Gamma_{l_q} \sum_{L_1 \dots L_{q-2}} (C_{l_1 0 \dots l_{q-2} 0 L_q 0}^{L_1 \dots L_{q-2} l; 0})^2. \quad (54)$$

We stress that the equalities in formulae (53) and (54) are consequences of (33). It will be also convenient to define a **-convolution* of order $p \geq 2$ as:

$$\begin{aligned} \widehat{\Gamma}_{p,l;l_1}^* &= \sum_{l_2} \dots \sum_{l_p} \Gamma_{l_2} \dots \Gamma_{l_p} \sum_{L_1 \dots L_{p-2}} \left\{ C_{l_1 0 l_2 0}^{L_1 0} C_{L_1 0 l_3 0}^{L_2 0} \dots C_{L_{p-2} 0 l_p 0}^{l 0} \right\}^2 \\ &= \sum_{l_2} \dots \sum_{l_p} \Gamma_{l_2} \dots \Gamma_{l_p} \sum_{L_1 \dots L_{p-2}} \left\{ C_{l_1 0 l_2 0 \dots l_p 0}^{L_1 \dots l; 0} \right\}^2. \end{aligned} \quad (55)$$

Note that the number of sums following the equalities in formula (55) is $p - 1$: however, we choose to keep the symbol p to denote $*$ -convolutions, since it is consistent with the probabilistic representations given in formulae (59) and (60) below. The above $*$ -convolution has the following property: for every $p = 2, \dots, q$

$$\sum_{l_1} \widehat{\Gamma}_{q+1-p, l_1} \widehat{\Gamma}_{p, l; l_1}^* = \widehat{\Gamma}_{q, l}, \text{ and, in particular, } \sum_{l_1} \Gamma_{l_1} \widehat{\Gamma}_{q, l; l_1}^* = \widehat{\Gamma}_{q, l}.$$

The $*$ -convolution of order 2 can be written more explicitly as

$$\widehat{\Gamma}_{2, l; l_1}^* = \sum_{l_2} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l_0})^2. \quad (56)$$

Remarks. (1) (*Probabilistic interpretation of the convolutions*) Write first $\Gamma_* \triangleq \sum_l \Gamma_l$ (plainly, in our framework $\Gamma_* = 4\pi$, but the following discussion applies to coefficients $\{\Gamma_l\}$ such that $\Gamma_* > 0$ is arbitrary) so that $l \mapsto \Gamma_l/\Gamma_*$ defines a probability on \mathbb{N} . The second orthonormality relation in (28) implies that, for fixed l_1, l_2 , the application $l \mapsto (C_{l_1 0 l_2 0}^{l_0})^2$ is a probability on \mathbb{N} . Now define the law of a (homogeneous) *Markov chain* $\{Z_n : n \geq 1\}$ as follows:

$$\mathbb{P}\{Z_1 = l\} = \Gamma_l/\Gamma_* \quad (57)$$

$$\mathbb{P}\{Z_{n+1} = l \mid Z_n = L\} = \sum_{l_0} \frac{\Gamma_{l_0}}{\Gamma_*} \left(C_{l_0 0 L 0}^{l_0}\right)^2. \quad (58)$$

It is clear that $\mathbb{P}\{Z_q = l\} = \widehat{\Gamma}_{q, l}/(\Gamma_*)^q$, and also, for $p \geq 2$,

$$\frac{\widehat{\Gamma}_{p, l; l_1}^*}{(\Gamma_*)^{p-1}} = \mathbb{P}\{Z_p = l \mid Z_1 = l_1\} \quad (59)$$

$$\frac{\widehat{\Gamma}_{p, l; l_1}^* \widehat{\Gamma}_{q+1-p, l_1}}{(\Gamma_*)^q} = \mathbb{P}\{(Z_q = l) \cap (Z_{q+1-p} = l_1)\} \quad (q > p - 1). \quad (60)$$

The following quantity will be crucial in the subsequent sections:

$$\frac{\widehat{\Gamma}_{q+1-p, l; \lambda}^* \widehat{\Gamma}_{p, \lambda}}{\sum_L \widehat{\Gamma}_{p, L} \widehat{\Gamma}_{q+1-p, l; L}^*} = \frac{\widehat{\Gamma}_{q+1-p, l; \lambda}^* \widehat{\Gamma}_{p, \lambda}}{\widehat{\Gamma}_{q, l}} = \mathbb{P}\{Z_p = \lambda \mid Z_q = l\} \quad (q > p); \quad (61)$$

observe that the last relation in (61) derives from

$$\widehat{\Gamma}_{q+1-p, l; \lambda}^* / (\Gamma_*)^{q-p} = \mathbb{P}\{(Z_{q+1-p} = l) \mid (Z_1 = \lambda)\} = \mathbb{P}\{(Z_q = \lambda) \mid (Z_p = l)\},$$

where the last equality is a consequence of the homogeneity of Z . Note also that we can identify each natural number $l \geq 0$ with an irreducible representation of $SO(3)$. It follows that the formal addition $l_1 + l_2 \triangleq \sum_l l (C_{l_1 0 l_2 0}^{l_0})^2$ may be used to endow $\widehat{SO(3)}$ with an hypergroup structure. In this sense, we can interpret the chain $\{Z_n : n \geq 1\}$ as a *random walk* on the hypergroup $\widehat{SO(3)}$, in a spirit similar to [8]. In Section 6, we will discuss a physical interpretation of these convolutions and establish a precise connection between the objects introduced in this section and the notion of convolution appearing in [8].

(2) (*A comparison with the Abelian case*) In [17], where we dealt with similar problems in the case of homogenous spaces of Abelian groups, we used extensively convolutions over \mathbb{Z} . This kind of convolutions, that we note ${}_A\widehat{\Gamma}_{q,l}$ ($q \geq 2, l \in \mathbb{Z}$) are obtained as in (52)-(56), by taking sums over \mathbb{Z} (instead than over \mathbb{N}) and by replacing the Clebsch-Gordan symbols $(C_{l_1 0 l_2 0}^{l 0})^2$ with the indicator $\mathbf{1}_{l_1+l_2=l}$. Note that these indicator functions do indeed provide the Clebsch-Gordan coefficients associated with the irreducible representations of the 1-dimensional torus $\mathbb{T} = [0, 2\pi)$, regarded as a compact Abelian group with group operation $xy = (x + y) \pmod{2\pi}$ (this is equivalent to the trivial relation $e^{il_1 x} e^{il_2 x} = \sum_l \mathbf{1}_{l_1+l_2=l} e^{il x} = e^{i(l_1+l_2)x}$). Note also that in the Abelian case one has ${}_A\widehat{\Gamma}_{p,l;l_1}^* = {}_A\widehat{\Gamma}_{p,l-l_1}$. Also, if $\Gamma_l = \Gamma_l^V$, where $\{\Gamma_l^V\}$ is the power spectrum of the Gaussian field V on \mathbb{T} appearing in (49), one has that $\widehat{{}_A\Gamma_{q,l}^V} = \mathbb{P}[U_q = l]$, where $\{U_n\}$ is the random walk given in (50).

5.2 The case $q = 2$

In this subsection, we provide a sufficient condition on the spectrum $\{C_l : l \geq 0\}$ (or, equivalently, on $\{\Gamma_l : l \geq 0\}$, as defined in (51)) to have the CLT (40) in the quadratic case $q = 2$. This condition is stated in Proposition 8, and is obtained via some preliminary (technical) computations and lemmas.

According to Part 3 of Theorem 3, to deal with (40) we shall find sufficient conditions to have that (39) takes place for $m = 0, q = 2$ and $p = 1$. From (37) we deduce

$$\widetilde{C}_l^{(2)} = 2 \left\{ \sum_{l_1, l_2=0}^{\infty} \frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)} C_{l_1} C_{l_2} (C_{l_1 0 l_2 0}^{l 0})^2 \right\}^2. \quad (62)$$

On the other hand, the multiple sums appearing in the numerator of (39) become ($q = 2, p = 1$)

$$\begin{aligned} & \left| \sum_{j_1, n_1, j_2, n_2} C_{j_1} C_{j_2} \sum_{l_1, m_1} C_{l_1} \mathcal{G}\{l_1, m_1; j_1, n_1; l, -m\} \mathcal{G}\{l_1, m_1; j_2, n_2; l, -m\} \right|^2 \\ &= \frac{1}{[4\pi(2l+1)]^2} \sum_{j_1, n_1, j_2, n_2} \Gamma_{j_1} \Gamma_{j_2} \left| \sum_{l_1, m_1} \Gamma_{l_1} C_{l_1 m_1 j_1 n_1}^{l m} C_{l_1 0 j_1 0}^{l 0} C_{l_1 m_1 j_2 n_2}^{l m} C_{l_1 0 j_2 0}^{l 0} \right|^2 \\ &= \frac{1}{[4\pi(2l+1)]^2} \sum_{j_1, n_1, j_2} \Gamma_{j_1} \Gamma_{j_2} \left| \sum_{l_1, m_1} \Gamma_{l_1} C_{l_1 m_1 j_1 n_1}^{l m} C_{l_1 0 j_1 0}^{l 0} C_{l_1 m_1 j_2 n_2}^{l m} C_{l_1 0 j_2 0}^{l 0} \right|^2 \\ &= \frac{1}{[4\pi(2l+1)]^2} \sum_{j_1 j_2} \sum_{l_1 l_2} \Gamma_{j_1} \Gamma_{j_2} \Gamma_{l_1} \Gamma_{l_2} C_{l_1 0 j_1 0}^{l 0} C_{l_1 0 j_2 0}^{l 0} C_{l_2 0 j_1 0}^{l 0} C_{l_2 0 j_2 0}^{l 0} \\ & \quad \times \left\{ \sum_{n_1 n_2 m_1 m_2} C_{l_1 m_1 j_1 n_1}^{l m} C_{l_1 m_1 j_2 n_2}^{l m} C_{l_2 m_2 j_1 n_1}^{l m} C_{l_2 m_2 j_2 n_2}^{l m} \right\}. \end{aligned} \quad (63)$$

Now, from [28, Eq. 8.7.4.20] we deduce that

$$\begin{aligned}
& \sum_{n_1 n_2} \sum_{m_1 m_2} C_{l_1 m_1 j_1 n_1}^{lm} C_{l_1 m_1 j_2 n_2}^{lm} C_{l_2 m_2 j_1 n_1}^{lm} C_{l_2 m_2 j_2 n_2}^{lm} \\
&= (-1)^\beta \sum_{s\sigma} (2s+1)(2l+1) (C_{lms\sigma}^{lm})^2 \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\} \\
&= \sum_s (2s+1)(2l+1) (C_{lms0}^{lm})^2 \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\},
\end{aligned}$$

where $\beta = l_1 + j_1 + l_2 + j_2$, and we used the Wigner $6j$ symbols, as defined in (35). The last equality follows because the quantity $l_1 + j_1 + l_2 + j_2 + 2l$ must be necessarily even, and therefore β must be even as well. It should be noted that the role of the pairs (j_1, n_1) and (l_1, m_1) is perfectly symmetric, so we obtain also

$$\begin{aligned}
& \sum_{n_1 n_2} \sum_{m_1 m_2} C_{l_1 m_1 j_1 n_1}^{lm} C_{l_1 m_1 j_2 n_2}^{lm} C_{l_2 m_2 j_1 n_1}^{lm} C_{l_2 m_2 j_2 n_2}^{lm} \\
&= \sum_s (2s+1)(2l+1) (C_{lms0}^{lm})^2 \left\{ \begin{matrix} j_1 & l_1 & l \\ l & s & j_2 \end{matrix} \right\} \left\{ \begin{matrix} j_1 & l_2 & l \\ l & s & j_2 \end{matrix} \right\},
\end{aligned}$$

whence

$$\sum_s (2s+1)(2l+1) (C_{lms0}^{lm})^2 \left\{ \begin{matrix} j_1 & l_1 & l \\ l & s & j_2 \end{matrix} \right\} \left\{ \begin{matrix} j_1 & l_2 & l \\ l & s & j_2 \end{matrix} \right\} \quad (64)$$

$$\equiv \sum_s (2s+1)(2l+1) (C_{lms0}^{lm})^2 \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\}. \quad (65)$$

Lemma 7 For all integers l, l_1, l_2, j_1, j_2 it holds that, for some positive constant c ,

$$\begin{aligned}
& \sum_s (2s+1)(2l+1) (C_{l0s0}^{l0})^2 \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\} \\
&\leq c \max \left[\frac{1}{\sqrt[5]{2l_1+1}} \wedge \frac{1}{\sqrt[5]{2l_2+1}}, \frac{1}{\sqrt[5]{2j_1+1}} \wedge \frac{1}{\sqrt[5]{2j_2+1}} \right].
\end{aligned}$$

Proof. Assume without loss of generality $j_1, j_2 > l_1$ otherwise we focus on (65) rather than (64). For $\alpha \in (0, 1)$, we have that

$$\begin{aligned}
& \sum_s (2s+1)(2l+1) (C_{l0s0}^{l0})^2 \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\} \\
&\leq \sum_{s \leq l_1^\alpha} (2s+1)(2l+1) (C_{l0s0}^{l0})^2 \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\} \\
&\quad + \sum_{s > l_1^\alpha} (2s+1)(2l+1) (C_{l0s0}^{l0})^2 \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq Cl_1^{2\alpha}(2l+1) \max_{s \leq l_1^\alpha} \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\} \\
&\quad + \left\{ \max_{s > l_1^\alpha} (C_{l_0 s 0}^{l_0})^2 \right\} \sum_s (2s+1)(2l+1) \left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \left\{ \begin{matrix} l_1 & j_2 & l \\ l & s & l_2 \end{matrix} \right\} \\
&\leq Cl_1^{2\alpha}(2l+1) \frac{1}{(2l+1)(2l_1+1)} + \frac{C}{l_1^{\alpha/2}} \frac{2l+1}{\sqrt{j_1 j_2}} = O(l_1^{2\alpha-1} + l_1^{-\alpha/2}) = O\left(\frac{1}{\sqrt[5]{l_1}}\right),
\end{aligned}$$

where the last equality has been obtained by setting $\alpha = 2/5$. The second last step follows because $j_1, j_2 \geq l_1, l_2$ implies $j_1, j_2 > l/2$, in view of the triangle inequalities $l_1 + j_1, l_1 + j_2 > l$; also, we used the inequality $\{\max_{s > l_1^\alpha} (C_{l_0 s 0}^{l_0})^2\} \leq l_1^{-\alpha/2}$, see Lemma 8 below. The bound with l_2 can be obtained by exploiting the symmetries of the $6j$ coefficients; in particular, we recall that (see ([28, Eq. 9.4.2.2]))

$$\left\{ \begin{matrix} l_1 & j_1 & l \\ l & s & l_2 \end{matrix} \right\} \equiv \left\{ \begin{matrix} l & j_1 & l_2 \\ l_1 & s & l \end{matrix} \right\} \equiv \left\{ \begin{matrix} l_2 & j_1 & l \\ l & s & l_1 \end{matrix} \right\}.$$

■

Remark. The bound provided in Lemma (7) is sufficient for our purposes below and we did not investigate its efficiency in detail. We remark, however, by setting $j_1 = j_2 = 0$, we have explicitly (see [28, Eq. 8.5.1.2])

$$\begin{aligned}
&\sum_{n_1 n_2 m_1 m_2} C_{l_1 m_1 j_1 n_1}^{lm} C_{l_1 m_1 j_2 n_2}^{lm} C_{l_2 m_2 j_1 n_1}^{lm} C_{l_2 m_2 j_2 n_2}^{lm} \\
&= \sum_{m_1 m_2} C_{l_1 m_1 00}^{lm} C_{l_1 m_1 00}^{lm} C_{l_2 m_2 00}^{lm} C_{l_2 m_2 00}^{lm} \equiv 1.
\end{aligned}$$

Lemma 8 As $l_1 \rightarrow +\infty$, $C_{l_0 l_1 0}^{l_0} = O(\frac{1}{\sqrt[4]{l_1}})$.

Proof. Unless the triangle condition $2l \geq l_1$ is satisfied, the Clebsch-Gordan coefficient is identically zero and the bound is trivial. Now recall that

$$C_{l_0 l_1 0}^{l_0} = \frac{\sqrt{2l+1} [(2l+l_1)/2]!}{[l_1/2]! [(2l-l_1)/2]! [l_1/2]!} \left\{ \frac{l_1! (2l-l_1)! l_1!}{(2l+l_1+1)!} \right\}^{1/2}.$$

For sequences $\{a_l\}$ and $\{b_l\}$, write $a_l \approx b_l$ when both $a_l = O(b_l)$ and $b_l = O(a_l)$ hold true. From Stirling's formula

$$\begin{aligned}
C_{l_0 l_1 0}^{l_0} &\approx \frac{\sqrt{2l+1} [(2l+l_1)/2]^{(2l+l_1)/2+1/2}}{[l_1/2]^{l_1+1} [(2l-l_1+1)/2]^{(2l-l_1)/2+1/2}} \left\{ \frac{l_1^{2l_1+1} (2l-l_1)^{(2l-l_1)+1/2}}{(2l+l_1+1)^{2l+l_1+3/2}} \right\}^{1/2} \\
&= \frac{\sqrt{2l+1} (2l+l_1)^{(2l+l_1)/2+1/2}}{l_1^{l_1+1} (2l-l_1+1)^{(2l-l_1)/2+1/2}} \left\{ \frac{l_1^{2l_1+1} (2l-l_1)^{(2l-l_1)+1/2}}{(2l+l_1+1)^{2l+l_1+3/2}} \right\}^{1/2} \\
&= \frac{\sqrt{2l+1}}{l_1^{1/2} (2l-l_1+1)^{1/4}} \frac{1}{(2l+l_1+1)^{1/4}} \leq \frac{\sqrt[4]{2l+1}}{l_1^{1/2} (2l-l_1+1)^{1/4}} = O\left(\frac{1}{\sqrt[4]{l_1}}\right)
\end{aligned}$$

■

We can finally state a sufficient condition for the CLT (40) in the case $q = 2$.

Proposition 9 For $q = 2$, a sufficient condition for the CLT (40) is the following asymptotic relation

$$\lim_{l \rightarrow +\infty} \sup_{l_1} \frac{\sum_{l_1} \Gamma_{l_1} \Gamma_{l_2} \{C_{l_1 0 l_2 0}^{l_0}\}^2}{\sum_{l_1, l_2} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l_0})^2} = \lim_{l \rightarrow +\infty} \sup_{l_1} \mathbb{P}\{Z_1 = l_1 \mid Z_2 = l_2\} = 0, \quad (66)$$

where the $\{\Gamma_l\}$ are given by (51) and $\{Z_l\}$ is the Markov chain defined in formulae (57) and (58).

Proof. In the sequel, we shall use repeatedly the trivial inequality

$$\sum_{j=0}^n \left| \frac{c_j}{a_j \wedge b_j} \right| = \sum_{j: a_j \leq b_j} \left| \frac{c_j}{a_j} \right| + \sum_{j: a_j > b_j} \left| \frac{c_j}{b_j} \right| \leq \sum_{j=0}^n \left| \frac{c_j}{a_j} \right| + \sum_{j=0}^n \left| \frac{c_j}{b_j} \right|, \quad (67)$$

which holds for arbitrary n and real vectors $\{a_j\}$, $\{b_j\}$ and $\{c_j\}$. In view of Lemma 7, by using a generalized Cauchy-Schwartz inequality, (67) and symmetry considerations, we obtain that the expression (63) is such that

$$\begin{aligned} (63) &\leq \frac{1}{[4\pi(2l+1)]^2} \sum_{j_1, j_2} \sum_{l_1, l_2} \Gamma_{j_1} \Gamma_{j_2} \Gamma_{l_1} \Gamma_{l_2} \left| C_{l_1 0 j_1 0}^{l_0} C_{l_1 0 j_2 0}^{l_0} C_{l_2 0 j_1 0}^{l_0} C_{l_2 0 j_2 0}^{l_0} \right| \\ &\leq \frac{2}{[4\pi(2l+1)]^2} \sum_{j_1, j_2} \sum_{l_1, l_2} \Gamma_{j_1} \Gamma_{j_2} \Gamma_{l_1} \Gamma_{l_2} \left| C_{l_1 0 j_1 0}^{l_0} C_{l_1 0 j_2 0}^{l_0} C_{l_2 0 j_1 0}^{l_0} C_{l_2 0 j_2 0}^{l_0} \right| \frac{1}{\sqrt[5]{j_1}} \\ &\leq \frac{1}{8[\pi(2l+1)]^2} \sqrt{\sum_{l_1 j_1} \frac{\Gamma_{l_1} \Gamma_{j_1}}{\sqrt[5]{j_1^2}} \{C_{l_1 0 j_1 0}^{l_0}\}^2 \sum_{l_1 j_2} \Gamma_{l_1} \Gamma_{j_2} \{C_{l_1 0 j_2 0}^{l_0}\}^2} \\ &\quad \times \sqrt{\sum_{l_2 j_1} \Gamma_{l_2} \Gamma_{j_1} \{C_{l_2 0 j_1 0}^{l_0}\}^2 \sum_{l_2 j_2} \Gamma_{l_2} \Gamma_{j_2} \{C_{l_2 0 j_2 0}^{l_0}\}^2}. \end{aligned}$$

The last expression is less than

$$\frac{\sum_{l_1 l_2} \Gamma_{l_1} \Gamma_{l_2} \{C_{l_1 0 l_2 0}^{l_0}\}^2}{8[\pi(2l+1)]^2} \sqrt{\sum_{l_1 j_1} \frac{\Gamma_{l_1} \Gamma_{j_1}}{\sqrt[5]{j_1^2}} \{C_{l_1 0 j_1 0}^{l_0}\}^2 \sum_{l_1 j_2} \Gamma_{l_1} \Gamma_{j_2} \{C_{l_1 0 j_2 0}^{l_0}\}^2}. \quad (68)$$

Now

$$\begin{aligned} \sum_{l_1 j_1} \frac{\Gamma_{l_1} \Gamma_{j_1}}{\sqrt[5]{j_1^2}} \{C_{l_1 0 j_1 0}^{l_0}\}^2 &\leq j^* \max_{j_1 \leq j^*} \left[\sum_{l_1 \geq 0} \Gamma_{l_1} \Gamma_{j_1} \{C_{l_1 0 j_1 0}^{l_0}\}^2 \right] \\ &\quad + \frac{1}{\sqrt[5]{(j^*)^2}} \sum_{l_1, j_1 \geq 0} \Gamma_{l_1} \Gamma_{j_1} \{C_{l_1 0 j_1 0}^{l_0}\}^2. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{(63)}{(62)} &\leq \frac{(68)}{\left\{ \sum_{l_1, l_2=0}^{\infty} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l_0})^2 \right\}^2} = \sqrt{\frac{\sum_{l_1 j_1} \frac{\Gamma_{l_1} \Gamma_{j_1}}{\sqrt[5]{j_1^2}} \{C_{l_1 0 j_1 0}^{l_0}\}^2}{\sum_{l_1, l_2=0}^{\infty} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l_0})^2}} \\ &\leq 2 \sqrt{\frac{j^* \max_{j_1 \leq j^*} \left[\sum_{l_1 \geq 1} \Gamma_{l_1} \Gamma_{j_1} \{C_{l_1 0 j_1 0}^{l_0}\}^2 \right]}{\sum_{l_1, l_2=0}^{\infty} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l_0})^2}} + \frac{1}{\sqrt[5]{(j^*)^2}}. \end{aligned}$$

Now fix $\varepsilon > 0$. Under (66) we have that, for any fixed and positive number $l_1^* > 1/\varepsilon$,

$$\begin{aligned} & \lim_{l \rightarrow \infty} \left[\frac{j^* \max_{j_1 \leq j^*} \left[\sum_{l_1 \geq 1} \Gamma_{l_1} \Gamma_{j_1} \left\{ C_{l_1 0 j_1 0}^{l_0} \right\}^2 \right]}{\sum_{l_1, l_2=1}^{\infty} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l_0})^2} + \frac{1}{\sqrt[5]{(j^*)^2}} \right] \\ & \leq j^* \lim_{l \rightarrow \infty} \sup_{l_1} \frac{\sum_{l_2=1}^{\infty} \Gamma_{l_1} \Gamma_{l_2} \left\{ C_{l_1 0 l_2 0}^{l_0} \right\}^2}{\sum_{l_1, l_2=1}^{\infty} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l_0})^2} + \sqrt[5]{\varepsilon^2} = \sqrt[5]{\varepsilon^2} . \end{aligned}$$

Because ε is arbitrary, the proof is concluded. ■

Remark. Note that, using (54) and (56), condition (66) becomes

$$\lim_{l \rightarrow \infty} \sup_{\lambda} \frac{\Gamma_{\lambda} \widehat{\Gamma}_{2,l;\lambda}^*}{\sum_{l_1} \Gamma_{l_1} \widehat{\Gamma}_{2,l;l_1}} = 0 . \quad (69)$$

Note also that if, in the convolutions (54), one replaces each squared Clebsch-Gordan coefficient $(C_{l_1 0 l_2 0}^{l_0})^2$ by the indicator $\mathbf{1}_{l_1+l_2=l}$ and extends the sums over \mathbb{Z} , one obtains the relation

$$\lim_{l \rightarrow \infty} \sup_{l_1} \frac{\Gamma_{l_1} \Gamma_{l-l_1}}{\sum_{l_1} \Gamma_{l_1} \Gamma_{l-l_1}} = 0. \quad (70)$$

In particular, when $\{\Gamma_l\} = \{\Gamma_l^V\}$ (the power spectrum of the field V on \mathbb{T} given in (49)) it is not difficult to show that formula (70) gives exactly the asymptotic (necessary and sufficient) condition (50).

5.3 The case $q = 3$

Our results for $q = 3$ closely mirrors the conditions we derived in the previous subsection.

Proposition 10 *A sufficient condition for the CLT (40) when $q = 3$ is*

$$\lim_{l \rightarrow \infty} \sup_{L_1} \frac{\sum_{l_1 l_2 j_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_1} \left\{ C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} \right\}^2}{\sum_{L_1} \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \left\{ C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0} \right\}^2} = 0, \text{ and} \quad (71)$$

$$\lim_{l \rightarrow \infty} \sup_{j_1} \frac{\sum_{l_1 l_2 L_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_1} \left\{ C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} \right\}^2}{\sum_{L_1} \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \left\{ C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0} \right\}^2} = 0 . \quad (72)$$

Remark. In the light of (54)-(56) and of the definition of the random walk Z given in (57) and (58), it is not difficult to see that (71) can be rewritten as

$$\begin{aligned} \lim_{l \rightarrow \infty} \sup_{\lambda} \frac{\widehat{\Gamma}_{2,\lambda} \sum_{j_1} \Gamma_{j_1} \left\{ C_{\lambda j_1 0}^{l_0} \right\}^2}{\widehat{\Gamma}_{3,l}} &= \lim_{l \rightarrow \infty} \sup_{\lambda} \frac{\widehat{\Gamma}_{2,\lambda} \widehat{\Gamma}_{2,l;\lambda}^*}{\sum_{L_1} \left[\widehat{\Gamma}_{2,L_1} \widehat{\Gamma}_{1,l;L_1}^* \right]} \\ &= \lim_{l \rightarrow \infty} \sup_{\lambda} \mathbb{P}[Z_2 = \lambda \mid Z_3 = l] = 0. \end{aligned} \quad (73)$$

Likewise, one obtains that (72) is equivalent to

$$\begin{aligned} & \lim_{l \rightarrow \infty} \sup_{j_1} \frac{\Gamma_{j_1} \hat{\Gamma}_{3,l;j_1}^*}{\sum_{L_1} \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \left\{ C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0} \right\}^2} \\ &= \lim_{l \rightarrow \infty} \sup_{j_1} \mathbb{P}[Z_1 = j_1 \mid Z_3 = l] = 0. \end{aligned} \quad (74)$$

It should be noted that the two conditions (73) and (74) can be written compactly as

$$\lim_{l \rightarrow \infty} \max_{q=1,2} \sup_{j_1} \frac{\hat{\Gamma}_{q,j_1} \hat{\Gamma}_{3-q,l;j_1}^*}{\sum_{L_1} \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \left\{ C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0} \right\}^2} = 0. \quad (75)$$

Relation (75) parallels once again analogous conditions established for stationary fields on a torus – see [17].

Proof of Proposition 10. In view of Part 3 of Theorem 3, we shall focus on the asymptotic negligibility of the ratio appearing in (39), in the case where $q = 3$ and $p = 2$. As before, the denominator of (39) is proportional to

$$\begin{aligned} & \left\{ \sum_{l_1, l_2, l_3} C_{l_1} C_{l_2} C_{l_3} \frac{1}{2l+1} \left\{ \prod_{i=1}^3 (2l_i + 1) \right\} \sum_{L_1} \left\{ C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0} \right\}^2 \right\}^2 \\ &= \frac{1}{(2l+1)^2} \left\{ \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \sum_{L_1} \left\{ C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0} \right\}^2 \right\}^2. \end{aligned} \quad (76)$$

On the other hand, the numerator is proportional to

$$\begin{aligned} & \frac{1}{(2l+1)^2} \sum_{j_1, j_2} \sum_{n_1, n_2} \Gamma_{j_1} \Gamma_{j_2} \times \\ & \left| \sum_{l_1, l_2, m_1, m_2} \Gamma_{l_1} \Gamma_{l_2} \sum_{L_1} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 m_1 l_2 m_2 j_1 n_1}^{L_1 l m} \sum_{L_2} C_{l_1 0 l_2 0 j_2 0}^{L_2 l_0} C_{l_1 m_1 l_2 m_2 j_2 n_2}^{L_2 l m} \right|^2 \\ &= \frac{1}{(2l+1)^2} \sum_{j_1, j_2} \sum_{n_1, n_2} \Gamma_{j_1} \Gamma_{j_2} \times \\ & \left| \sum_{l_1, l_2, m_1, m_2} \Gamma_{l_1} \Gamma_{l_2} \sum_{L_1} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} \sum_{M_1} C_{l_1 m_1 l_2 m_2}^{L_1 M_1} C_{L_1 M_1 j_1 n_1}^{l m} \right. \\ & \quad \left. \sum_{L_2} C_{l_1 0 l_2 0 j_2 0}^{L_2 l_0} \sum_{M_2} C_{l_1 m_1 l_2 m_2}^{L_2 M_2} C_{L_2 M_2 j_2 n_2}^{l m} \right|^2 \end{aligned} \quad (77)$$

This last expression equals in turn

$$\begin{aligned}
&= \frac{1}{(2l+1)^2} \sum_{j_1, j_2} \sum_{n_1, n_2} \Gamma_{j_1} \Gamma_{j_2} \times \\
&\quad \left| \sum_{l_1, l_2} \Gamma_{l_1} \Gamma_{l_2} \sum_{L_1} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} \sum_{M_1} C_{L_1 M_1 j_1 n_1}^{lm} \sum_{L_2} C_{l_1 0 l_2 0 j_2 0}^{L_2 l_0} \sum_{M_2} C_{L_2 M_2 j_2 n_2}^{lm} \delta_{L_1}^{L_2} \delta_{M_1}^{M_2} \right|^2 \\
&= \frac{1}{(2l+1)^2} \sum_{j_1, j_2} \sum_{n_1, n_{21}} \Gamma_{j_1} \Gamma_{j_2} \times \\
&\quad \left| \sum_{l_1, l_2=0} \Gamma_{l_1} \Gamma_{l_2} \sum_{L_1} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} \sum_{M_1} C_{L_1 M_1 j_1 n_1}^{lm} C_{L_1 M_1 j_2 n_2}^{lm} \right|^2 \\
&= \frac{1}{(2l+1)^2} \sum_{j_1, j_2} \sum_{n_1, n_{21}} \Gamma_{j_1} \Gamma_{j_2} \times \\
&\quad \left| \sum_{l_1, l_2} \Gamma_{l_1} \Gamma_{l_2} \sum_{L_1} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} \sum_{M_1} C_{L_1 M_1 j_1 n_1}^{lm} \sum_{L_2} C_{l_1 0 l_2 0 j_2 0}^{L_2 l_0} \sum_{M_2} C_{L_2 M_2 j_2 n_2}^{lm} \delta_{L_1}^{L_2} \delta_{M_1}^{M_2} \right|^2
\end{aligned}$$

and we can use the same argument as for $q = 2$. More precisely, one can write

$$\begin{aligned}
&\left| \sum_{l_1, l_2=1} \Gamma_{l_1} \Gamma_{l_2} \sum_{L_1} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} \sum_{M_1} C_{L_1 M_1 j_1 n_1}^{lm} C_{L_1 M_1 j_2 n_2}^{lm} \right|^2 \\
&= \sum_{l_1 \dots l_4} \Gamma_{l_1} \dots \Gamma_{l_4} \sum_{L_1 L_2} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0} C_{l_3 0 l_4 0 j_2 0}^{L_2 l_0} \\
&\quad \sum_{M_1 M_2} C_{L_1 M_1 j_1 n_1}^{lm} C_{L_1 M_1 j_2 n_2}^{lm} C_{L_2 M_2 j_1 n_1}^{lm} C_{L_2 M_2 j_2 n_2}^{lm} \\
&= \sum_{l_1 \dots l_4} \Gamma_{l_1} \dots \Gamma_{l_4} \sum_{L_1 L_2} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0} C_{l_3 0 l_4 0 j_2 0}^{L_2 l_0} \\
&\quad (-1)^\zeta \sum_{s\sigma} (2s+1)(2l+1) (C_{l_0 s \sigma}^{l_0})^2 \left\{ \begin{matrix} L_1 & j_1 & l \\ l & s & L_2 \end{matrix} \right\} \left\{ \begin{matrix} L_1 & j_2 & l \\ l & s & L_2 \end{matrix} \right\} \quad (78)
\end{aligned}$$

where $\zeta = L_1 + j_1 + L_2 + j_2$, and (78) equals

$$\begin{aligned}
&= \sum_{l_1 \dots l_4} \Gamma_{l_1} \dots \Gamma_{l_4} \sum_{L_1 L_2} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0} C_{l_3 0 l_4 0 j_2 0}^{L_2 l_0} \\
&\quad (-1)^{2l} \sum_s (2s+1)(2l+1) (C_{l m s 0}^{lm})^2 \left\{ \begin{matrix} L_1 & j_1 & l \\ l & s & L_2 \end{matrix} \right\} \left\{ \begin{matrix} L_1 & j_2 & l \\ l & s & L_2 \end{matrix} \right\}.
\end{aligned}$$

From (7) we now obtain that the last expression is bounded by

$$\begin{aligned}
&\sum_{l_1 \dots l_4} \Gamma_{l_1} \dots \Gamma_{l_4} \sum_{L_1 L_2} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0} C_{l_3 0 l_4 0 j_2 0}^{L_2 l_0} \frac{1}{\sqrt[5]{L_1}} \\
&+ \sum_{l_1 \dots l_4} \Gamma_{l_1} \dots \Gamma_{l_4} \sum_{L_1 L_2} C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0} C_{l_3 0 l_4 0 j_2 0}^{L_2 l_0} \frac{1}{\sqrt[5]{j_1}},
\end{aligned}$$

whence all the terms are bounded by

$$\sum_{j_1 j_2} \sum_{l_1 \dots l_4} \sum_{L_1 L_2} \Gamma_{j_1} \Gamma_{j_2} \Gamma_{l_1} \dots \Gamma_{l_4} C_{l_1 0 l_2 0}^{L_1 0} C_{L_1 0 j_1 0}^{l_0} C_{l_1 0 l_2 0}^{L_1 0} \times \quad (79)$$

$$\times C_{L_1 0 j_2 0}^{l_0} C_{l_3 0 l_4 0}^{L_2 0} C_{L_2 0 j_1 0}^{l_0} C_{l_3 0 l_4 0}^{L_2 0} C_{L_2 0 j_2 0}^{l_0} \frac{1}{\sqrt[5]{L_1}}$$

$$+ \sum_{j_1 j_2} \sum_{l_1 \dots l_4} \sum_{L_1 L_2} \Gamma_{j_1} \Gamma_{j_2} \Gamma_{l_1} \dots \Gamma_{l_4} C_{l_1 0 l_2 0}^{L_1 0} C_{L_1 0 j_1 0}^{l_0} C_{l_1 0 l_2 0}^{L_1 0} \times \quad (80)$$

$$\times C_{L_1 0 j_2 0}^{l_0} C_{l_3 0 l_4 0}^{L_2 0} C_{L_2 0 j_1 0}^{l_0} C_{l_3 0 l_4 0}^{L_2 0} C_{L_2 0 j_2 0}^{l_0} \frac{1}{\sqrt[5]{j_1}}.$$

Also,

$$\sum_{j_1 j_2} \sum_{l_1 \dots l_4} \Gamma_{j_1} \Gamma_{j_2} \Gamma_{l_1} \dots \Gamma_{l_4} \sum_{L_1 L_2} \frac{C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0} C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0} C_{l_3 0 l_4 0 j_2 0}^{L_2 l_0}}{\sqrt[5]{L_1}}$$

$$= \sum_{\substack{j_1 j_2 \\ l_1 \dots l_4}} \Gamma_{j_1} \Gamma_{j_2} \Gamma_{l_1} \dots \Gamma_{l_4} \sum_{L_1 \dots L_4} \frac{C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0} C_{l_1 0 l_2 0 j_2 0}^{L_3 l_0} C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0} C_{l_3 0 l_4 0 j_2 0}^{L_4 l_0}}{\sqrt[5]{L_1}} \delta_{L_1}^{L_3} \delta_{L_2}^{L_4}$$

$$\leq \sqrt{\sum_{l_1 l_2 j_1 L_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_1} \frac{\{C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0}\}^2}{L_1^{2/5}}} \sqrt{\sum_{l_1 l_2 j_2 L_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_2} \{C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0}\}^2} \times$$

$$\sqrt{\sum_{l_3 l_4 j_1 L_2} \Gamma_{l_3} \Gamma_{l_4} \Gamma_{j_1} \{C_{l_3 0 l_4 0 j_1 0}^{L_2 l_0}\}^2} \sqrt{\sum_{l_3 l_4 j_2 L_2} \Gamma_{l_3} \Gamma_{l_4} \Gamma_{j_2} \{C_{l_3 0 l_4 0 j_2 0}^{L_2 l_0}\}^2}$$

$$\leq \sqrt{\sum_{l_1 l_2 j_1 L_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_1} \frac{\{C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0}\}^2}{L_1^{2/5}}} \left\{ \sum_{l_1 l_2 j_2 L_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_2} \{C_{l_1 0 l_2 0 j_2 0}^{L_1 l_0}\}^2 \right\}^{3/2}. \quad (81)$$

To sum up, we have obtained

$$\frac{(79)}{(76)} \leq \frac{\frac{1}{(2l+1)^2} \times (81)}{\frac{1}{(2l+1)^2} \left\{ 6 \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \sum_{L_1} \{C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0}\}^2 \right\}^2}$$

$$\leq \left[\frac{\sum_{l_1 l_2 j_1 L_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_1} \{C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0}\}^2 / L_1^{2/5}}{6 \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \sum_{L_1} \{C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0}\}^2} \right]^{1/2}, \quad (82)$$

By an identical argument we obtain also

$$\frac{(80)}{(76)} \leq \left[\frac{\sum_{l_1 l_2 j_1 L_1} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{j_1} \{C_{l_1 0 l_2 0 j_1 0}^{L_1 l_0}\}^2 / j_1^{2/5}}{6 \sum_{l_1, l_2, l_3} \Gamma_{l_1} \Gamma_{l_2} \Gamma_{l_3} \sum_{L_1} \{C_{l_1 0 l_2 0 l_3 0}^{L_1 l_0}\}^2} \right]^{1/2}. \quad (83)$$

Now we can adopt exactly the same line of reasoning as in the proof of Proposition 9, so that by trivial manipulations we deduce that (71) and (72) are indeed sufficient to have that the RHS of (82) and (83) converges to zero as $l \rightarrow +\infty$. \square

5.4 The case of a general q : results and conjectures

The following proposition gives a general version of the results proved in Sections 5.2 and 5.3. The proof (omitted) is rather long, and can be obtained along the lines of those of Proposition 9 and Proposition 10.

Proposition 11 *Fix $q \geq 4$. Then, a sufficient condition to have the asymptotic relation (39) in the case $p = q - 1$ is the following;*

$$\begin{aligned} & \lim_{l \rightarrow \infty} \left\{ \sup_{\lambda} \frac{\widehat{\Gamma}_{q-1,\lambda} \widehat{\Gamma}_{2,l;\lambda}^*}{\sum_L \widehat{\Gamma}_{q-1,L} \widehat{\Gamma}_{1,l;L}^*} + \sup_{\lambda} \frac{\widehat{\Gamma}_{q,l;\lambda}^* \Gamma_{\lambda}}{\sum_L \widehat{\Gamma}_{q-1,L} \widehat{\Gamma}_{1,l;L}^*} \right\} \\ &= \lim_{l \rightarrow \infty} \left\{ \sup_{\lambda} \frac{\widehat{\Gamma}_{q-1,\lambda} \widehat{\Gamma}_{2,l;\lambda}^*}{\widehat{\Gamma}_{q,l}} + \sup_{\lambda} \frac{\widehat{\Gamma}_{q,l;\lambda}^* \Gamma_{\lambda}}{\widehat{\Gamma}_{q,l}} \right\} = 0. \end{aligned} \quad (84)$$

Remarks. (1) As in the proofs of Proposition 9 and Proposition 10, a crucial technique in proving Proposition 11 consists in the simplification of sums of the type

$$\sum_{\substack{m_1 m_2 m_3 \\ M_1 \dots M_4}} C_{l_1 m_1 l_2 m_2}^{L_1 M_1} C_{L_1 M_1 l_3 m_3}^{L_2 M_2} C_{L_2 M_2 j_1 n_1}^{l m} C_{l_1 m_1 l_2 m_2}^{L_3 M_3} C_{L_3 M_3 l_3 m_3}^{L_4 M_4} C_{L_4 M_4 j_2 n_2}^{l m 2}, \quad (85)$$

by means of the general relation

$$\sum_{m_1 m_2} C_{l_1 m_1 l_2 m_2}^{L_1 M_1} C_{l_1 m_1 l_2 m_2}^{L_3 M_3} = \delta_{L_1}^{L_3} \delta_{M_1}^{M_3}. \quad (86)$$

This basically means that, if in (85) each Clebsch-Gordan coefficient is represented as the vertex of a connected graph, then it is possible to “reduce” such graph by cutting edges corresponding to 2-loops – see [14] for a more detailed discussion on these graphical methods.

(2) Note that, since $q \geq 4$ and according to Part C of Theorem 3, condition (39) is *only necessary* to have the CLT (40), so that (84) cannot be used to deduce the asymptotic Gaussianity of the frequency components of Hermite-subordinated fields of the type $H_q[T]$. Some conjectures concerning the case $q \geq 4$, $p \neq q - 1$ are presented at the end of the section.

(3) Observe that, in terms of the random walk $\{Z_n\}$ defined in (57)-(58),

$$\frac{\widehat{\Gamma}_{q-1,\lambda} \widehat{\Gamma}_{2,l;\lambda}^*}{\widehat{\Gamma}_{q,l}} = \mathbb{P}\{Z_{q-1} = \lambda \mid Z_q = l\} \quad \text{and} \quad \frac{\widehat{\Gamma}_{q,l;\lambda}^* \Gamma_{\lambda}}{\widehat{\Gamma}_{q,l}} = \mathbb{P}\{Z_1 = \lambda \mid Z_q = l\}.$$

As mentioned before, the relation (39) (which implies (40)), in the general case where $q \geq 4$ and $p \neq q - 1$, is still being investigated, as it requires a hard analysis of higher order Clebsch-Gordan coefficients by means of graphical techniques (see for instance [28, Ch. 11]). At this stage, it is however natural to propose the following conjecture. Recall that we focus on the CLT (40) because of the equality in law $T_l^{(q)}(x) = \sqrt{\frac{2l+1}{4\pi}} a_{l0;q}$, and Corollary 4.

Conjecture A (Weak) *A sufficient condition for the CLT (40) is*

$$\begin{aligned} & \lim_{l \rightarrow \infty} \max_{1 \leq p \leq q-1} \sup_{\lambda} \frac{\widehat{\Gamma}_{p,\lambda} \widehat{\Gamma}_{q+1-p,l;\lambda}^*}{\sum_L \widehat{\Gamma}_{p,L} \widehat{\Gamma}_{q+1-p,l;L}^*} \\ &= \lim_{l \rightarrow \infty} \max_{1 \leq p \leq q-1} \sup_{\lambda} \mathbb{P}\{Z_p = \lambda \mid Z_q = l\} = 0. \end{aligned} \quad (87)$$

It is worth emphasizing how condition (87) is the exact analogous of the necessary and sufficient condition (50), established in [17] for the high-frequency CLT on the torus $\mathbb{T} = [0, 2\pi)$. This remarkable circumstance may suggest the following (much more general and, for the time being, quite imprecise) extension.

Conjecture B (Strong) *Let T be an isotropic Gaussian field defined on the homogeneous space of a compact group G , and set $T^{(q)} = H_q(T)$ ($q \geq 2$). Then, the high-frequency components of $T^{(q)}$ are asymptotically Gaussian if, and only if, it holds a condition of the type*

$$\lim_{l \rightarrow l_0} \max_{1 \leq p \leq q-1} \sup_{\lambda \in \widehat{G}} \frac{\widehat{\Gamma}_{p,\lambda}^* \widehat{\Gamma}_{q+1-p,l;\lambda}}{\sum_{L \in \widehat{G}} \widehat{\Gamma}_{p,L}^* \widehat{\Gamma}_{q+1-p,l;L}} = 0, \quad (88)$$

where \widehat{G} is the dual of G , l_0 is some point at the boundary of \widehat{G} , and the convolutions $\widehat{\Gamma}$ and $\widehat{\Gamma}^*$ are defined (analogously to (52)-(55)) on the power spectrum of T , by means of the appropriate Clebsch-Gordan coefficients of the group.

We leave the two Conjectures A and B as open issues for future research.

Remark. (On “no privileged path” conditions) In terms of Z , condition (87) can be further interpreted as follows: for every l , define a “bridge” of length q , by conditioning Z to equal l at time q . Then, (87) is verified if, and only if, the probability that the bridge hits λ at time q converges to zero, uniformly on λ , as $l \rightarrow +\infty$. It is also evident that, when (87) is verified for every $p = 1, \dots, q-1$, one also has that

$$\lim_{l \rightarrow +\infty} \sup_{\lambda_1, \dots, \lambda_{q-1} \in \mathbb{N}} \mathbb{P}[Z_1 = \lambda_1, \dots, Z_{q-1} = \lambda_{q-1} \mid Z_q = l] = 0, \quad (89)$$

meaning that, asymptotically, the law of Z does not charge any “privileged path” of length q leading to l . The interpretation of condition (89) in terms of bridges can be reinforced by putting by convention $Z_0 = 0$, so that the probability in (89) is that of the particular path $0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_{q-1} \rightarrow l$, associated with a random bridge linking 0 and l .

6 Further physical interpretation of the convolutions and connection with other random walks on hypergroups

6.1 Convolutions as mixed states

We recall that, in quantum mechanics, it is customary to consider two possible initial states for a particle, i.e. those provided by the so-called *pure states*, where the state of a particle is given, and those provided by the so-called *mixed states*, where the state of the particle is given by a mixture (in the usual probabilistic sense) over different quantum states. We refer the reader to [13] for an introduction to these ideas. From this standpoint, the quantity $\widehat{\Gamma}_{q,l}$ defined in (54) is the probability associated to a mixed state, where the mixing is performed over all possible values of the total angular momentum. To illustrate this point, we use the standard bra-ket notation $|l0\rangle$ to indicate the state of a particle having total angular momentum equal to l and projection 0 on the z -axis. By using this formalism, the quantity $\widehat{\Gamma}_{q,l}$ can be obtained as follows:

- (i) consider a system of q particles $\alpha_1, \dots, \alpha_q$ such that each α_j is in the mixed state Ξ according to which a particle is in the state $|k0\rangle$ with probability Γ_k/Γ_* ($k \geq 0$);

- (ii) obtain $\widehat{\Gamma}_{q,l}$ as the probability that the elements of this system are coupled pairwise to form a particle in the state $|l0\rangle$.

Now denote by $\mathbf{A}_{p,|\lambda 0\rangle}$ the event that the first p particles $\alpha_1, \dots, \alpha_p$ have coupled pairwise to generate the state $|\lambda 0\rangle$. Then, one also has that

$$\frac{\widehat{\Gamma}_{p+1,\lambda} \widehat{\Gamma}_{q-p,l;\lambda}^*}{\widehat{\Gamma}_{q,l}} = \Pr \{ \text{the } q \text{ particles generate } |l0\rangle \mid \mathbf{A}_{p,|\lambda 0\rangle} \}. \quad (90)$$

In particular, relation (90) yields a further physical interpretation of the “no privileged path condition” discussed in (89).

6.2 Other convolutions and random walks on group duals

Random walks on hypergroups, and specifically on group duals, have been actively studied in the seventies – see [8, Ch. 6]. Our aim in the sequel is to compare our definitions with those provided in this earlier literature, mainly by discussing the alternative physical meanings of the associated notion of convolution. We recall from Section 3 that, starting from the Wigner’s D -matrices representation of $SO(3)$, we obtain the unitary equivalent reducible representations $\{D^{l_1}(g) \otimes D^{l_2}(g)\}$ and $\{\oplus_{l=|l_2-l_1|}^{l_2+l_1} D^l(g)\}$. Now note $\chi_l(g)$ the character of $D^l(g)$; for all $g \in SO(3)$, we have immediately

$$\chi_{l_1}(g) \chi_{l_2}(g) = \sum_{l=|l_2-l_1|}^{l_2+l_1} \chi_l(g) .$$

In [8, p. 222], an alternative class of Clebsch-Gordan coefficients $\{C_{l_1 l_2 | G}^l : l_1, l_2, l \geq 0\}$ is defined by means of the identity

$$\frac{1}{2l_1+1} \chi_{l_1}(g) \frac{1}{2l_2+1} \chi_{l_2}(g) = \sum_l C_{l_1 l_2 | G}^l \frac{1}{2l+1} \chi_l(g)$$

which leads to

$$C_{l_1 l_2 | G}^l = \frac{2l+1}{(2l_1+1)(2l_2+1)} \{l_1 l_2 l\} ,$$

where we use the same notation as in [28] and in many other physical textbooks, i.e. we take $\{l_1 l_2 l\}$ to represent the indicator function of the event $|l_2 - l_1| \leq l \leq l_2 + l_1$. Of course

$$C_{l_1 l_2 | G}^l = \sum_{l=|l_2-l_1|}^{l_2+l_1} \frac{2l+1}{(2l_1+1)(2l_2+1)} \equiv 1 . \quad (91)$$

As observed in [8], relation (91) can be used to endow $\widehat{SO(3)}$ with an hypergroup structure, via the formal addition $l_1 + l_2 \triangleq \sum_l l C_{l_1 l_2 | G}^l$. Now let $\{\Gamma_l : l \geq 0\}$ be a collection of positive coefficients such that $\sum_l \Gamma_l = 1$. The convolutions and *-convolutions of the $\{\Gamma_l\}$ that are naturally associated with the above formal addition are given by

$$\widetilde{\Gamma}_{2,l} = \sum_{l_1, l_2} \Gamma_{l_1} \Gamma_{l_2} C_{l_1 l_2 | G}^l , \quad \widetilde{\Gamma}_{3,l} = \sum_{L_1, l_3} \widetilde{\Gamma}_{2, L_1} \Gamma_{l_3} C_{L_1 l_3 | G}^l , \dots \quad (92)$$

$$\widetilde{\Gamma}_{q,l} = \sum_{L_1, l_q} \widetilde{\Gamma}_{q-1, L_{q-1}} \Gamma_{l_q} C_{L_{q-1} l_q | G}^l , \quad (93)$$

and, for $p \geq 2$,

$$\tilde{\Gamma}_{p,l;l_1}^* = \sum_{l_2} \cdots \sum_{l_p} \Gamma_{l_2} \cdots \Gamma_{l_p} \sum_{L_1 \dots L_{p-2}} C_{l_1 l_2 | G}^{L_1} C_{L_1 l_3 | G}^{L_2} \cdots C_{L_{p-2} l_p | G}^l. \quad (94)$$

As shown in [8], the objects appearing in (92)-(94) can be used to define the law of a random walk $\tilde{Z} = \{\tilde{Z}_n : n \geq 1\}$ on \mathbb{N} (regarded as an hypergroup isomorphic to $\widehat{SO(3)}$), exactly as we did in (57)-(58). In particular, since $\Gamma_* = \sum \Gamma_l = 1$, one has that $\tilde{\Gamma}_{p,l;l_1}^* = \mathbb{P}\{\tilde{Z}_p = l \mid \tilde{Z}_1 = l_1\}$. Also, the convolutions (92)-(94) (and therefore the random walk \tilde{Z}) enjoy a physical interpretation which is interesting to compare with our previous result. To see this, assume we have two mixed states Ξ_{l_1} and Ξ_{l_2} : in state Ξ_{l_1} , the particle has total angular momentum l_1 and its projection on the axis z takes values $m_1 = -l_1, \dots, l_1$ with uniform (classical) probability $(2l_1 + 1)^{-1}$; analogous conditions are imposed for Ξ_{l_2} . Let us now compute the probability $\Pr\{l \mid \Xi_{l_1}, \Xi_{l_2}\}$ that the system will couple to form a particle with total angular momentum l and arbitrary projection on z . Start by observing that the probability that a particle in the state $|l_1 m_1\rangle$ will couple with another particle in the state $|l_2 m_2\rangle$ to yield the state $|lm\rangle$ is exactly given by $\{C_{l_1 m_1 l_2 m_2}^{lm}\}^2$. Hence, with straightforward notation,

$$\begin{aligned} \Pr\{l \mid \Xi_{l_1}, \Xi_{l_2}\} &= \sum_{m_1 m_2} \Pr\{l \mid |l_1 m_1\rangle, |l_2 m_2\rangle\} \Pr\{m_1, m_2\} \\ &= \sum_{m_1 m_2} \Pr\{l \mid |l_1 m_1\rangle, |l_2 m_2\rangle\} \frac{1}{2l_1 + 1} \frac{1}{2l_2 + 1} \\ &= \sum_m \sum_{m_1 m_2} \{C_{l_1 m_1 l_2 m_2}^{lm}\}^2 \frac{1}{2l_1 + 1} \frac{1}{2l_2 + 1} \\ &= \sum_m \frac{\{l_1 l_2 l\}}{2l_1 + 1} \frac{1}{2l_2 + 1} = \frac{2l + 1}{2l_1 + 1} \frac{\{l_1 l_2 l\}}{2l_2 + 1} = C_{l_1 l_2 | G}^l. \end{aligned} \quad (95)$$

It follows from (95) that the quantity $\tilde{\Gamma}_{q,l}$ can be obtained as follows:

- (i) consider a system of q particles $\alpha_1, \dots, \alpha_q$ such that each α_j is in the mixed state Ξ according to which a particle is in the state $|ku\rangle$, $u = -k, \dots, k$, with probability $(2k + 1)^{-1} \Gamma_k / \Gamma_*$ ($k \geq 0$);
- (ii) obtain $\tilde{\Gamma}_{q,l}$ as the probability that the elements of this system are coupled pairwise to form a particle in the state $|lm\rangle$, any $m = -l, \dots, l$.

To sum up, both convolutions $\hat{\Gamma}$ and $\tilde{\Gamma}$ can be interpreted in terms of random interacting quantum particles: $\hat{\Gamma}$ -type convolutions are obtained from particles in mixed states where the mixing is performed over pure states of the form $|k0\rangle$; on the other hand, $\tilde{\Gamma}$ -type convolutions are associated with mixed state particles where mixing is over pure states of the type $\{|ku\rangle : u = -k, \dots, k\}$, uniformly in u for every fixed k .

7 Application: algebraic/exponential dualities

In this section we discuss explicit conditions on the angular power spectrum $\{C_l : l \geq 0\}$ of the Gaussian field T introduced in Section 2, ensuring that the CLT (40) may hold. Our results

show that, if the power spectrum decreases exponentially, then a high-frequency CLT holds, whereas the opposite implication holds if the spectrum decreases as a negative power. This duality mirrors analogous conditions previously established in the Abelian case – see [17]. For simplicity, we stick to the case $q = 2$. Note that the results below allow to deal with the asymptotic (high-frequency) behaviour of the Sachs-Wolfe model (6).

7.1 The Exponential case

Assume

$$C_l \approx (l+1)^\alpha \exp(-l), \quad \alpha \geq 0. \quad (96)$$

To prove that, in this case, (40) is verified for $q = 2$, we will prove that (66) holds (recall the definition of Γ_l given in (51)). For the denominator of the previous expression we obtain the lower bound

$$\begin{aligned} \sum_{l_1, l_2=1}^{\infty} \Gamma_{l_1} \Gamma_{l_2} (C_{l_1 0 l_2 0}^{l 0})^2 &\geq \sum_{l_1=[l/3]}^{[2l/3]} \Gamma_{l_1} \Gamma_{l-l_1} (C_{l_1 0 l-l_1 0}^{l 0})^2 \\ &\approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} (C_{l_1 0 l-l_1 0}^{l 0})^2 \end{aligned} \quad (97)$$

and in view of [28], equation 8.5.2.33, and Stirling's formula

$$\begin{aligned} (97) &\approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} \left(\frac{l!}{l_1! (l-l_1)!} \right)^2 \left(\frac{(2l_1)! (2l-2l_1)!}{(2l)!} \right) \\ &\approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} \frac{l^{2l+1}}{l_1^{2l_1+1} (l-l_1)^{2l-2l_1+1}} \\ &\quad \times \left(\frac{(2l_1)^{2l_1+1/2} (2l-2l_1)^{2l-2l_1+1/2}}{(2l)^{2l+1/2}} \right) \\ &\approx \exp(-l) l^{2(\alpha+1)} \sum_{l_1=[l/3]}^{[2l/3]} \frac{l^{1/2}}{l_1^{1/2} (l-l_1)^{1/2}} \approx \exp(-l) l^{2(\alpha+1)} l^{1/2}. \end{aligned}$$

On the other hand, recall that by the triangle conditions (Section 3) $\{C_{l_1 0 l_2 0}^{l 0}\}^2 \equiv 0$ unless $l_1 + l_2 \geq l$. Hence

$$\begin{aligned} \sup_{l_1} \sum_{l_2} \Gamma_{l_1} \Gamma_{l_2} \left\{ C_{l_1 0 l_2 0}^{l 0} \right\}^2 &\leq K \sup_{l_1} \exp(-l) l_1^{\alpha+1} \\ &\times \left\{ |l-l_1|^{\alpha+1} + \sum_{u=1}^{\infty} \exp(-u) |l_1+u|^{\alpha+1} \right\} \approx \exp(-l) l^{2(\alpha+1)} l^{1/2}. \end{aligned}$$

It is then immediate to see that that (66) is satisfied.

7.2 Regularly varying functions

For $q = 2$, we show below that the CLT fails for all sequences C_l such that: (a) C_l is quasi monotonic, i.e. $C_{l+1} \leq C_l(1 + K/l)$, and (b) C_l is such that $\liminf_{l \rightarrow \infty} C_l/C_{l/2} > 0$. In particular, a necessary condition for the CLT (40) to hold is that $C_l/C_{l/2} \rightarrow 0$. This is exactly the same necessary condition as was derived by [17] in the Abelian case. For the general case $q \geq 2$, we expect the CLT fails for all regularly varying angular power spectra, i.e. for all C_l such that $\liminf_{\ell \rightarrow \infty} C_l/C_{\alpha\ell} > 0$ for all $\alpha > 0$. Note that we are thus covering all polynomial forms for C_l^{-1} .

Since (66) only provides a sufficient condition for the CLT, we need to analyze directly the more primitive condition (39) for $m = 0$ (however, the case $m \neq 0$ entails just a more complicated notation). We consider first an upper bound for the square root of the denominator of (39), which is given by $\tilde{C}_l^{(2)}$.

We have

$$\begin{aligned} \tilde{C}_l^{(2)} &= \sum_{j_1, j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{4\pi(2l+1)} \left(C_{j_1 0 j_2 0}^{l0} \right)^2 \\ &\leq 2 \sum_{j_1, j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{4\pi(2l+1)} \left(C_{j_1 0 j_2 0}^{l0} \right)^2 \\ &= \frac{1}{2\pi} \sum_{j_1} C_{j_1} (2j_1+1) \sum_{j_2=j_1}^{\infty} C_{j_2} \left(C_{j_1 0 l 0}^{j_2 0} \right)^2 \\ &\leq \frac{1}{2\pi} \sum_{j_1} C_{j_1} (2j_1+1) \left\{ \sup_{j_2 \geq j_1, j_1+j_2 > l} C_{j_2} \right\} \sum_{j_2=0}^{\infty} \left(C_{j_1 0 l 0}^{j_2 0} \right)^2 \leq K C_{l/2}. \end{aligned}$$

where we have used the relation $\frac{2j_2+1}{2l+1} (C_{j_1 0 j_2 0}^{l0})^2 = (C_{j_1 0 l 0}^{j_2 0})^2$, as well as

$$\sup_{j_2 \geq j_1, j_1+j_2 > l} C_{j_2} \leq K C_{l/2}, \text{ and } \sum_{l=|l_2-l_1|}^{l_2+l_1} \left(C_{l_1 0 l_2 0}^{l0} \right)^2 \equiv 1.$$

For the numerator of (39) one has that it is greater than

$$\begin{aligned} &\sum_{j_1, j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{(4\pi(2l+1))^2} \left| \sum_{l_1} C_{l_1} (2l_1+1) C_{l_1 0 j_1 0}^{l0} C_{l_1 0 j_1 0}^{l0} C_{l_1 0 j_2 0}^{l0} C_{l_1 0 j_2 0}^{l0} \right|^2 \\ &\geq \sum_{j_1, j_2} C_{j_1} C_{j_2} \frac{(2j_1+1)(2j_2+1)}{(4\pi(2l+1))^2} \left| 5 C_2 \left\{ C_{20 j_1 0}^{l0} C_{20 j_2 0}^{l0} \right\}^2 \right|^2 \\ &\geq C_l^2 \frac{1}{(4\pi)^2} \left| 5 C_2 \left\{ C_{20 l 0}^{l0} \right\}^2 \right|^2 \geq K C_l^2. \end{aligned}$$

The left-hand side of condition (39) is then bounded below by $\lim_{l \rightarrow \infty} (K_1 C_l^2) / (K_2 C_{l/2}^2) \neq 0$, so that the CLT (40) cannot hold.

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